Sharif Quantum Information Group

Topological Quantum Computation-Part I

Objectives

- To understand the basic ideas of:

Topological Qubit

Topological Order

Kitaev Model

Topological Quantum Computation

Classical Bits

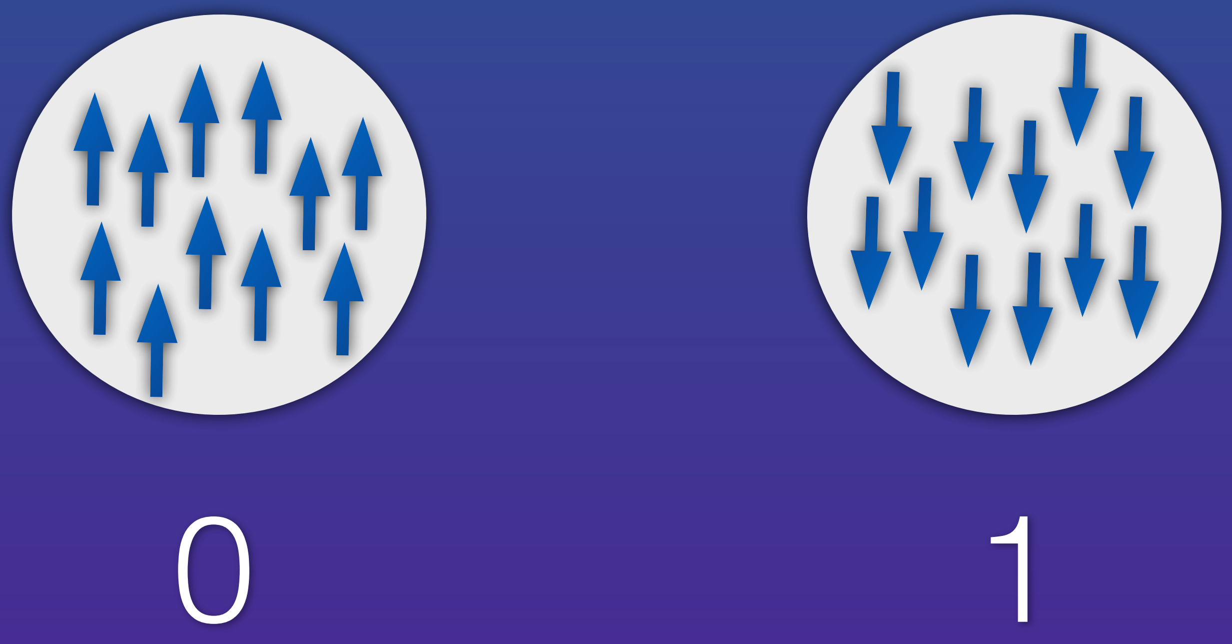
Quantum Bits

Classical Bits Have

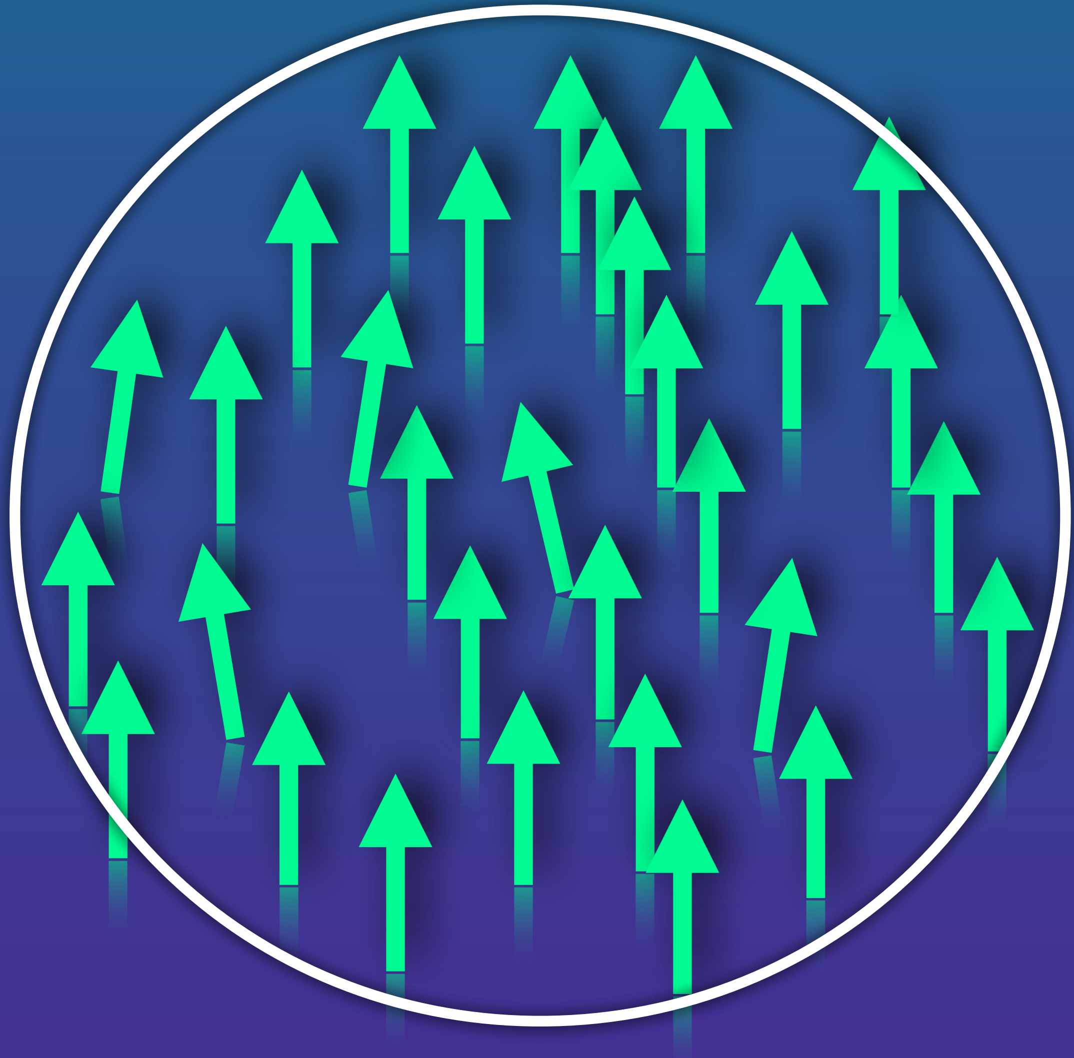
Four

Very Good Properties

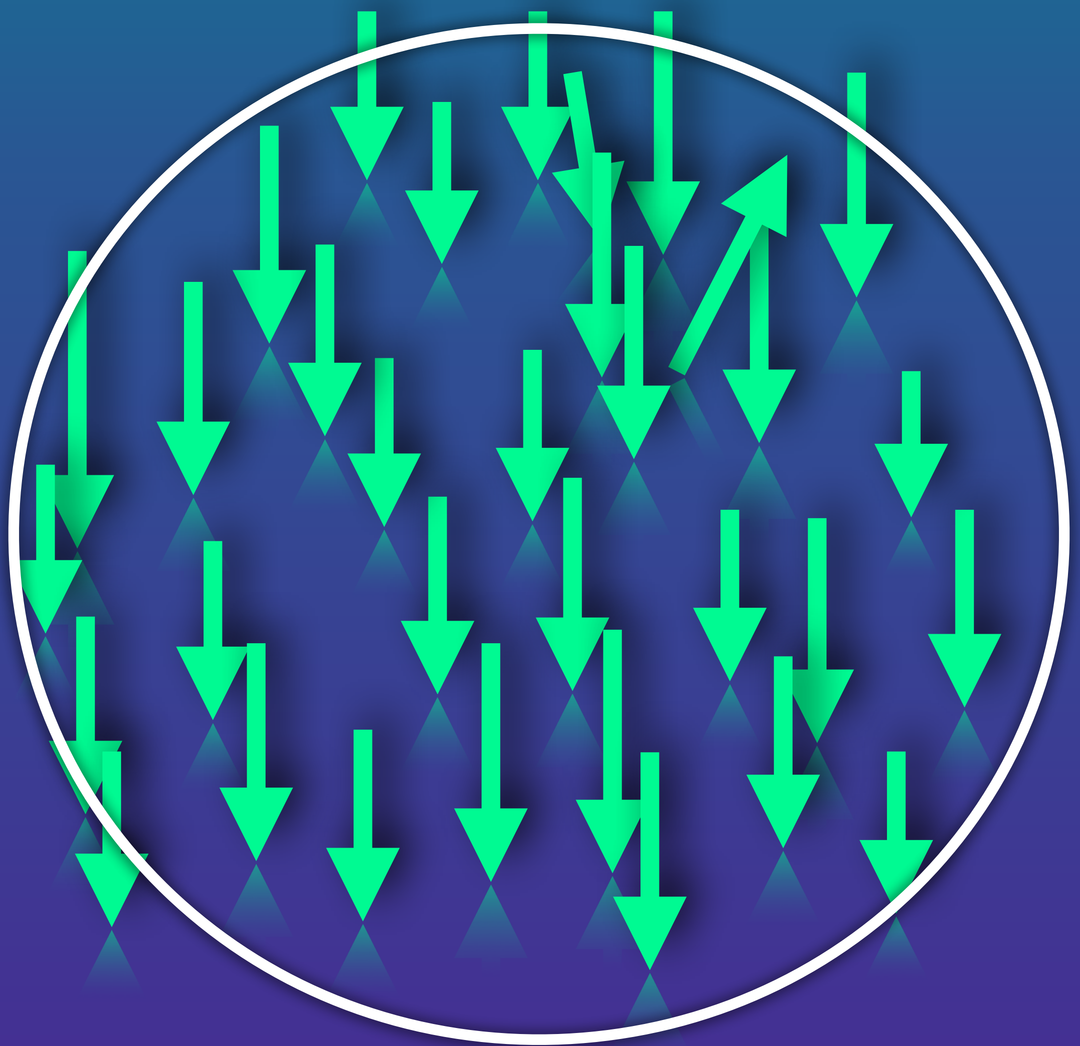
1- Bits are Macroscopic Objects



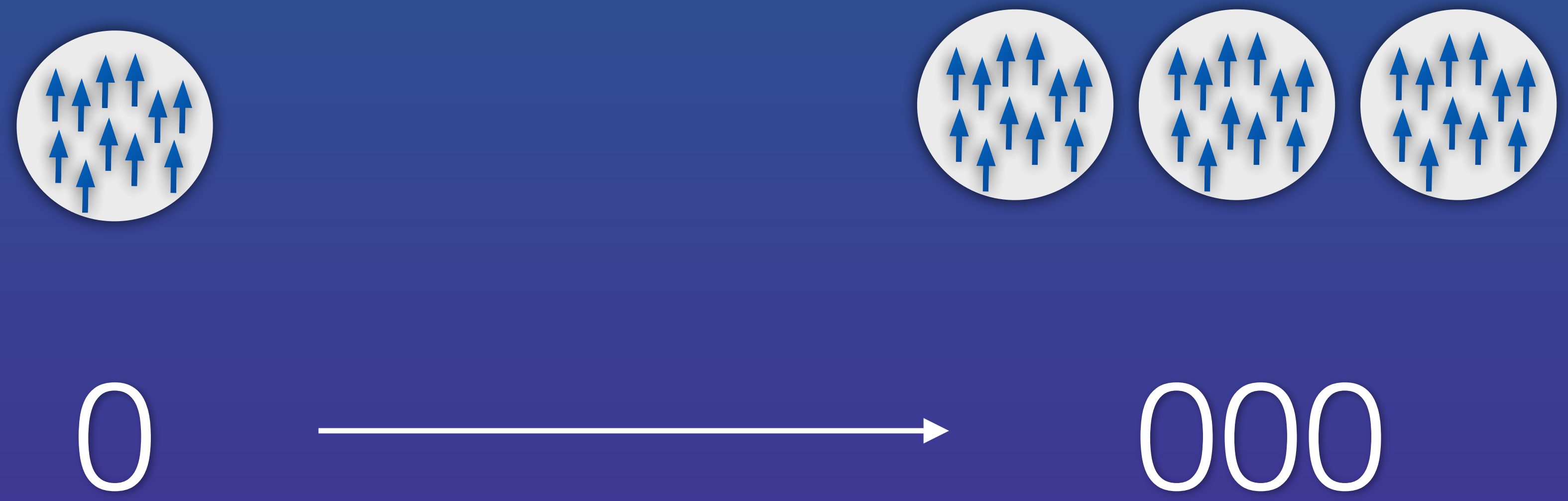
0



1



2- Bits can be cloned



3- Errors are discrete

0 ————————> 1

1 ————————> 0

4- Bits can be observed

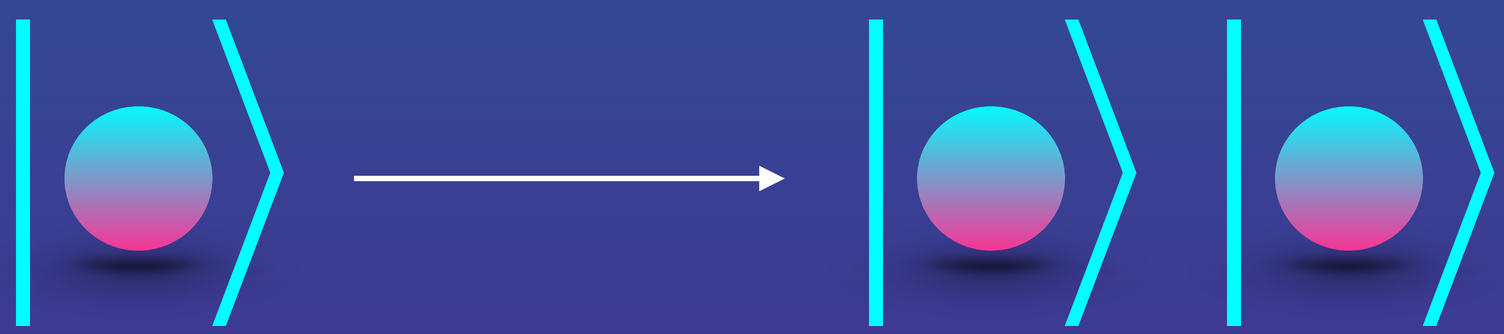
010 → 000

And corrected

Qubits are exactly the opposite

They are microscopic

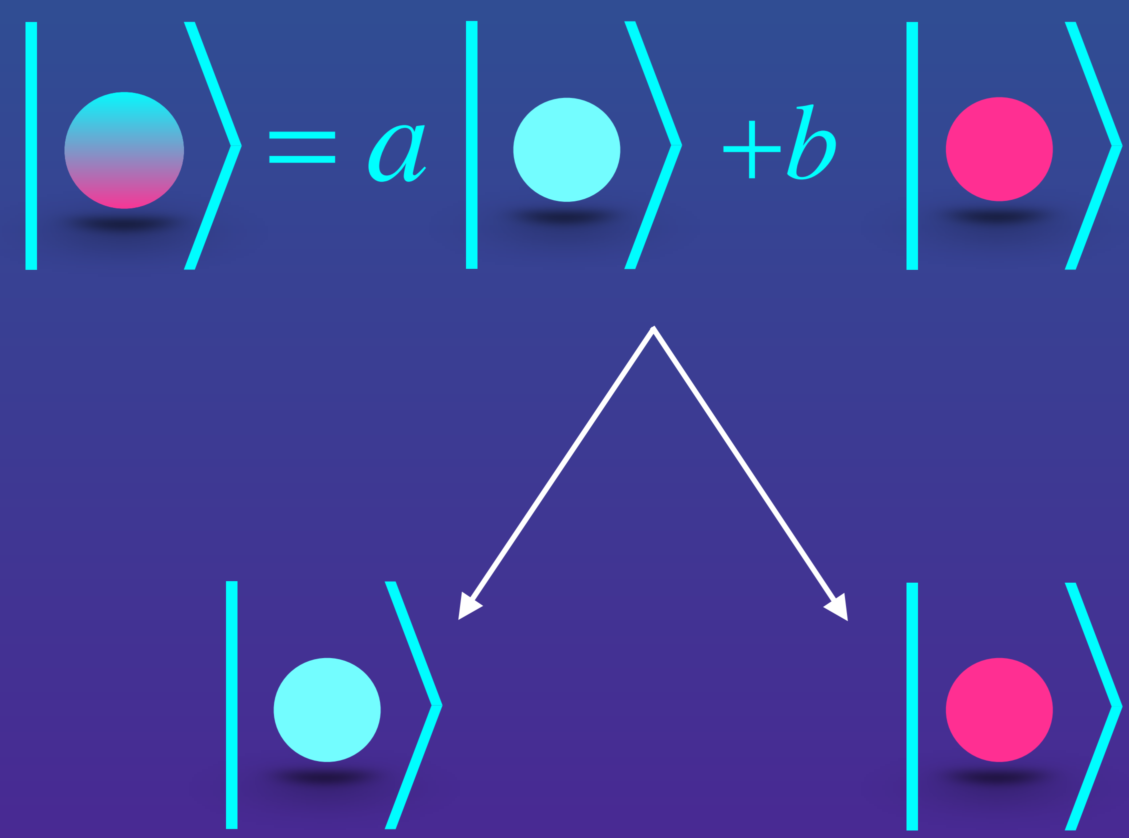
They cannot be cloned



$$|\psi\rangle \otimes |0\rangle \rightarrow |\psi\rangle \otimes |\psi\rangle$$

Quantum Errors are continuous

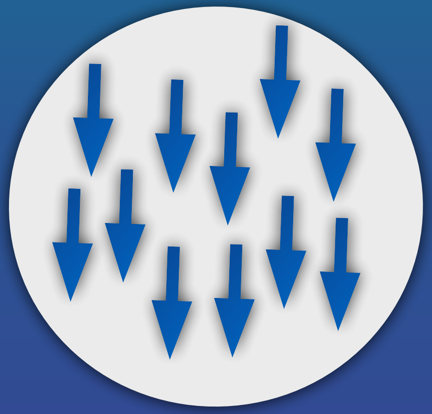
They cannot be observed



Topological Qubits

Merging the good features of both

$|\bar{0}\rangle$



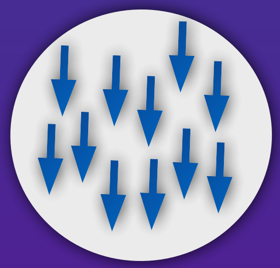
$|\bar{1}\rangle$

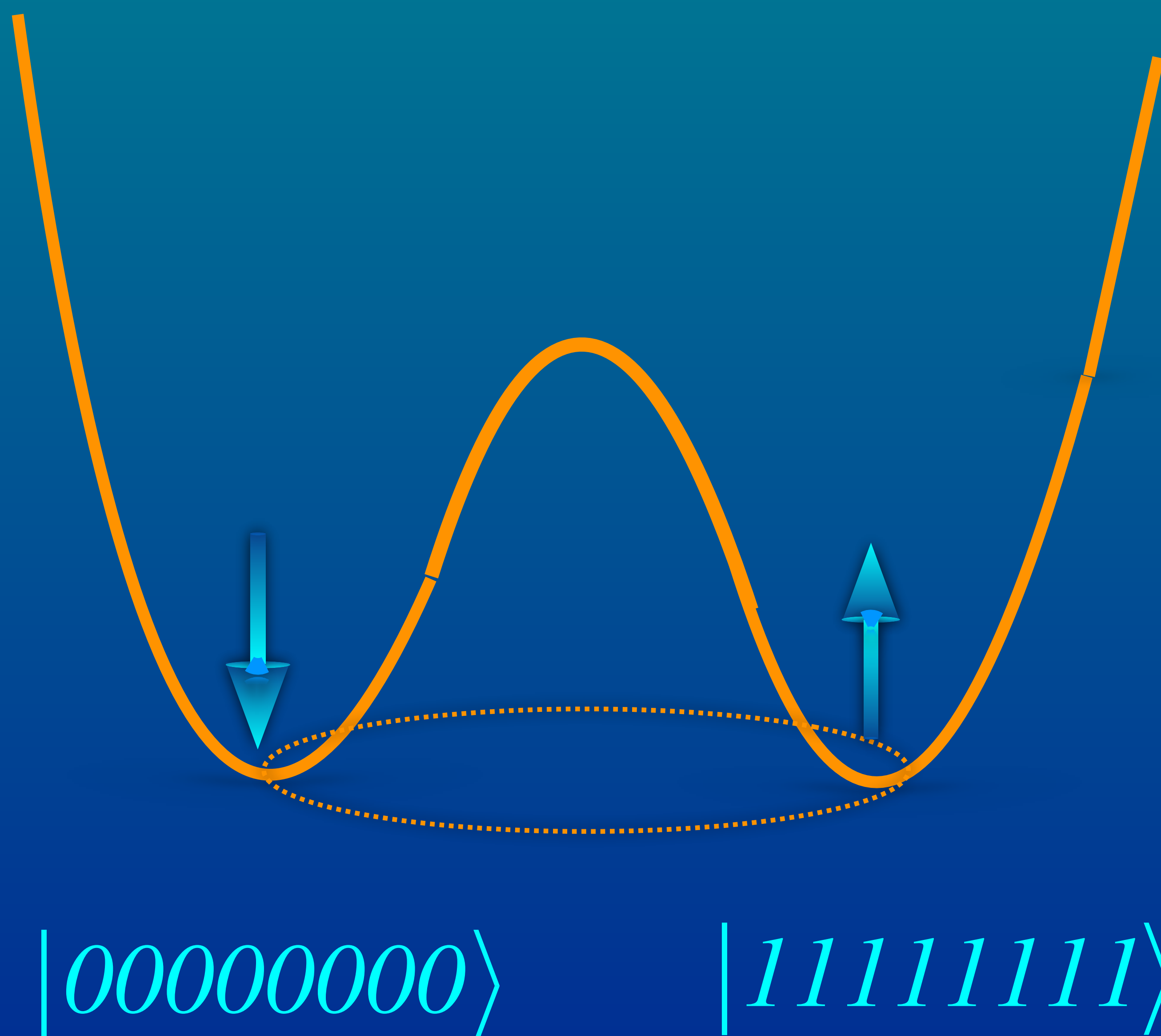
$$a|\bar{0}\rangle + b|\bar{1}\rangle$$

Ising Model

$$H = -\sum_{<i,j>} z_i z_j$$

$$|\bar{0}\rangle = |\uparrow\uparrow\uparrow \ldots \uparrow\uparrow\uparrow\rangle \qquad |\bar{1}\rangle = |\downarrow\downarrow\downarrow \ldots \downarrow\downarrow\downarrow\rangle$$

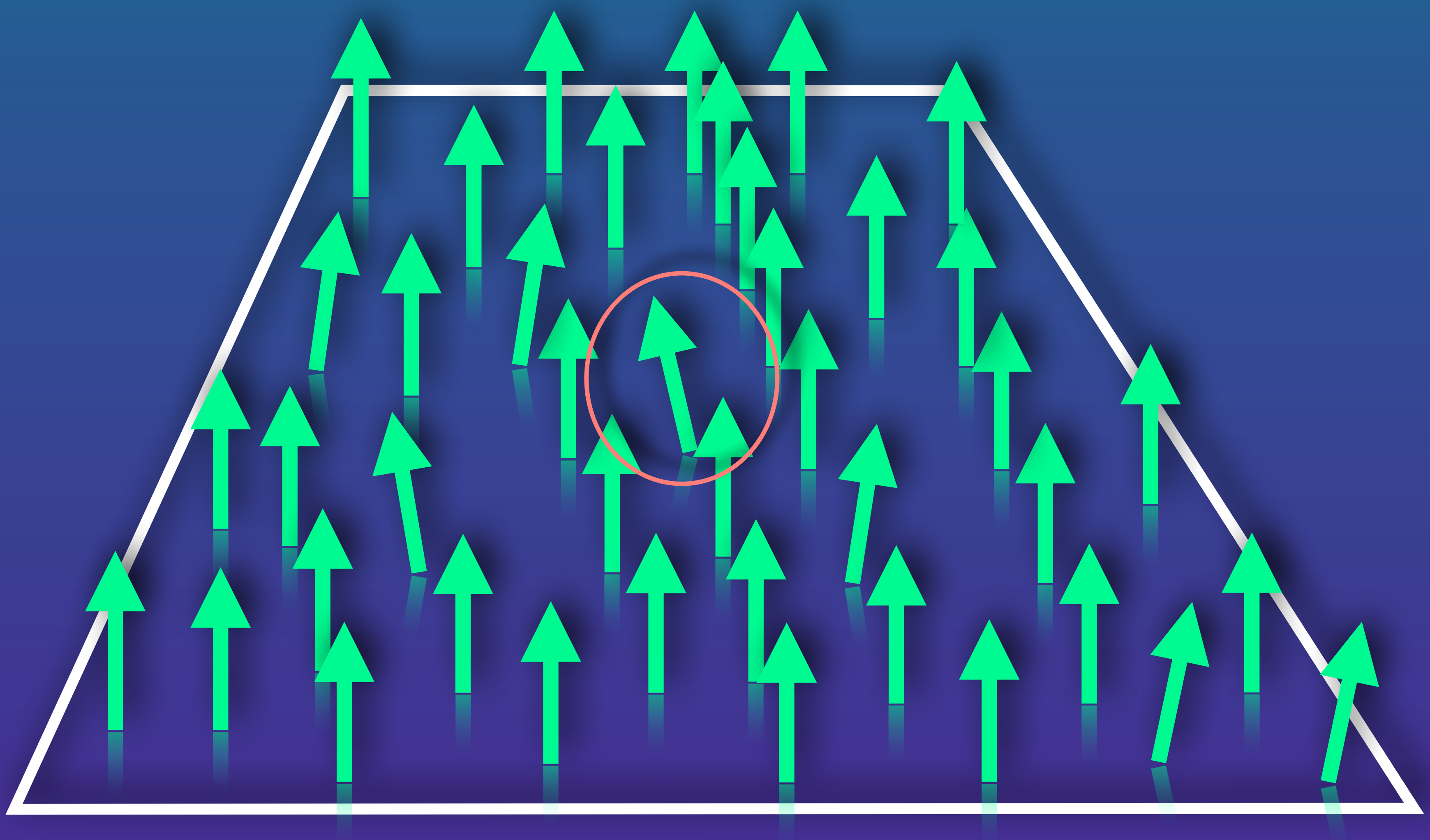



|00000000⟩
|11111111⟩

Local Order

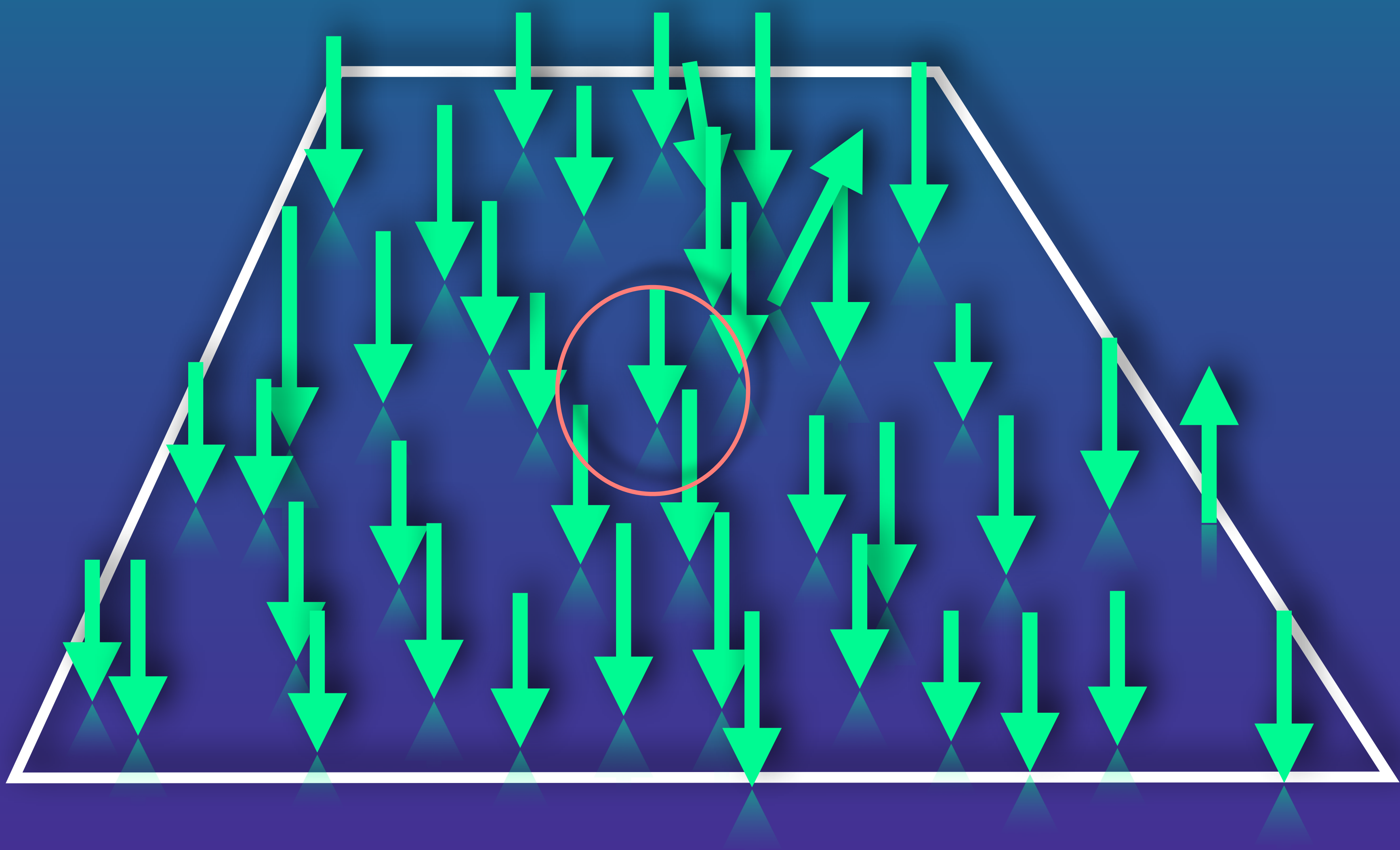
$\langle \sigma_z \rangle = 1$

$|0\rangle$



$$\langle \sigma_z \rangle = -1$$

$$|1\rangle$$



But local order cannot produce a topological qubit!

Local order is extremely fragile

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|0000\rangle + |1111\rangle\right)$$

$$|0000\rangle \qquad\qquad |1111\rangle$$

$$|W\rangle = \frac{1}{2}\left(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle\right)$$

$$\frac{1}{4} \qquad\qquad \frac{3}{4}$$

$$|W_1\rangle = |1000\rangle \qquad\qquad |W_0\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$$

$$|W\rangle = \frac{1}{\sqrt{N}}\left(|100....000\rangle + |010....000\rangle + |001....000\rangle + ... + |00....001\rangle\right)$$

$$\frac{N-1}{N}$$

$$|W_0\rangle = \frac{1}{\sqrt{N-1}}\left(|100....00\rangle + |010....00\rangle + |001....00\rangle + ... + |00....01\rangle\right)$$

What we want?

Degenerate ground state

Existence of Gap

Not locally distinguishable

Robust to perturbations

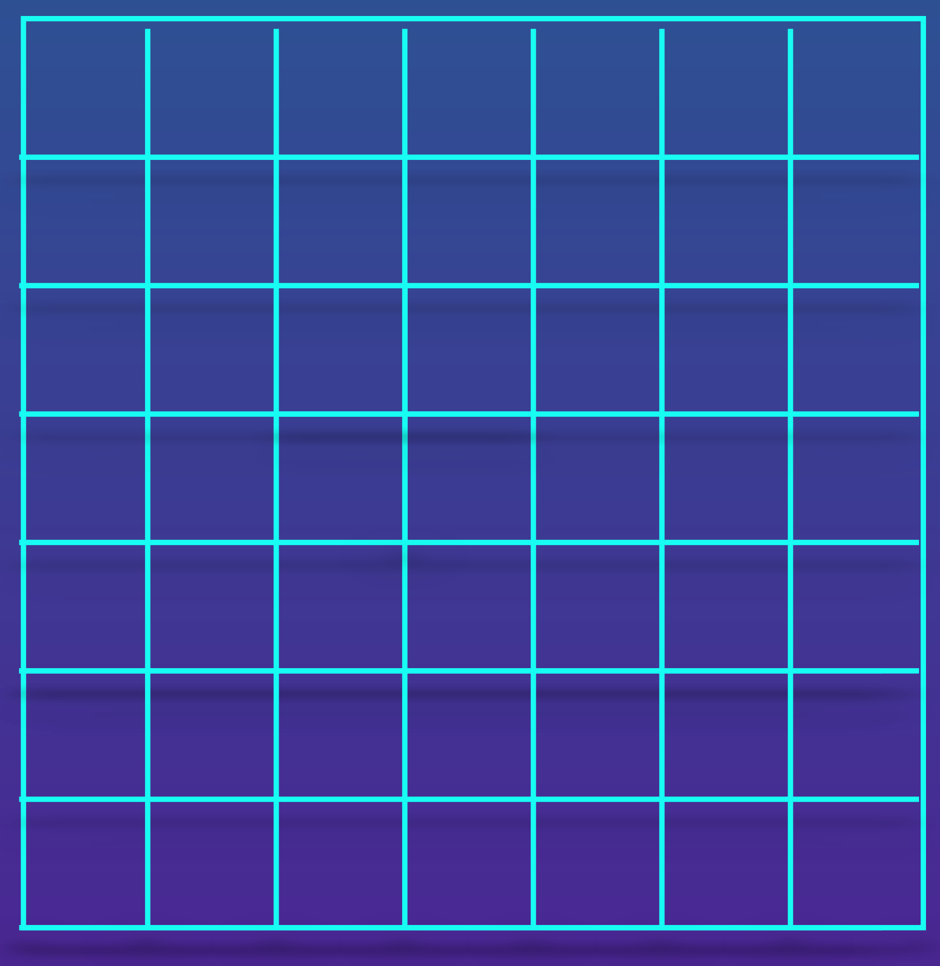
A system with degenerate ground state

$|\psi_0\rangle$ $|\psi_1\rangle$

which cannot be distinguished,
by any local observable!

$$\langle\psi_0|K|\psi_0\rangle = \langle\psi_1|K|\psi_1\rangle$$

How to make a topological model



$$K_i^2 = \pm I$$

$$K_i \left| \psi_\alpha \right\rangle = \left| \psi_\alpha \right\rangle$$

Local Operator

A ground state

$$\left\langle \psi_\alpha \right| K_i \left| \psi_\alpha \right\rangle = 1$$

$$K_i^2 = I$$

Stabilizers

$$\left[K_i \ , \ K_j \right] = 0$$

The Hamiltonian

$$H = -K_1 - K_2 - \ldots\ldots - K_M$$

All the local operators

$$H\left|\psi_\alpha\right\rangle = -M\left|\psi_\alpha\right\rangle$$

A ground state

The order of degeneracy

$$Dim(H) = 2^N$$

$$Degeneracy = \frac{2^N}{2^M} = 2^{N-M}$$

Kitaev Model

Notations

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \left\{ \begin{array}{l} |+\rangle \\ |-\rangle \end{array} \right.$$

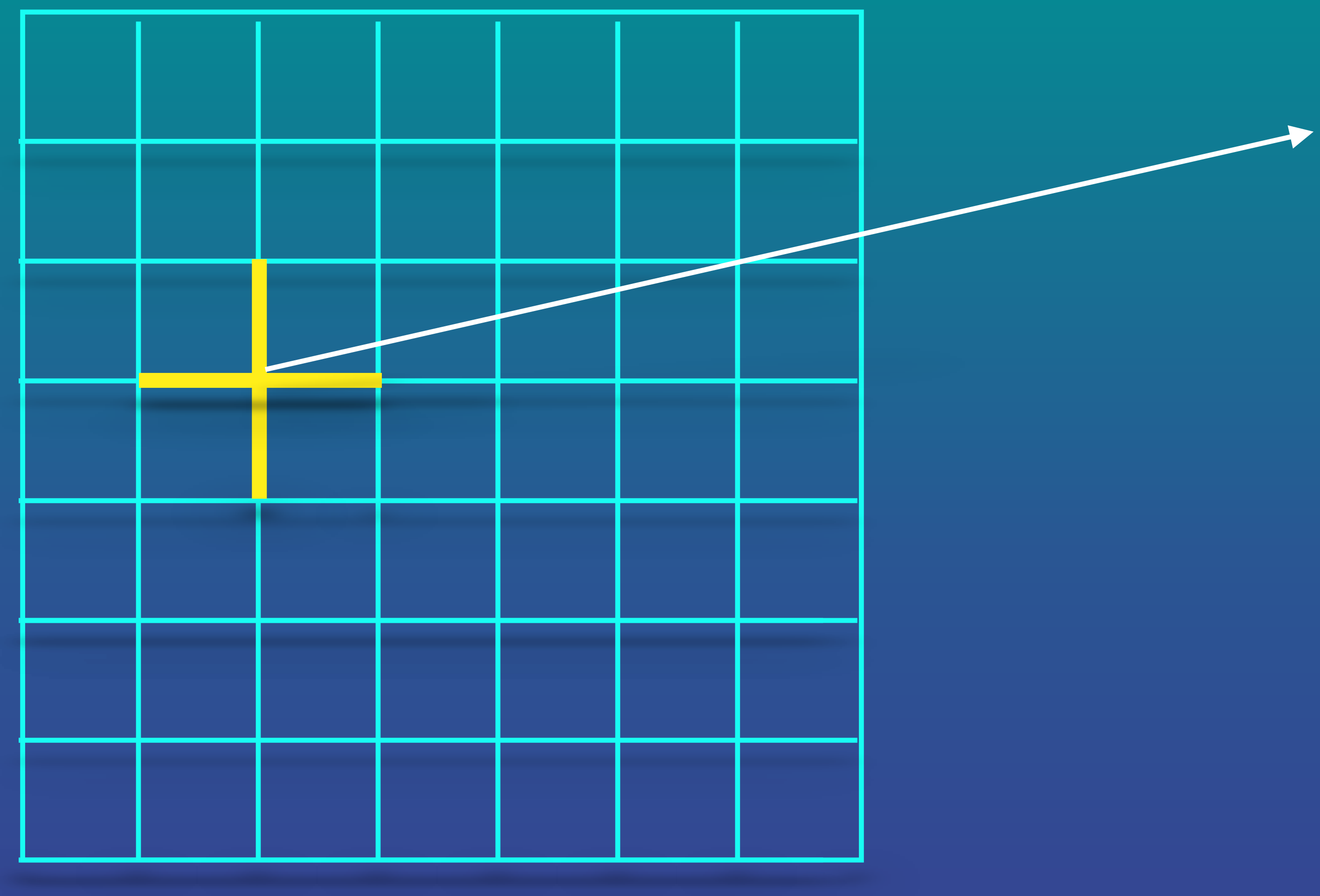
$$z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \left\{ \begin{array}{l} |0\rangle \\ |1\rangle \end{array} \right.$$

$$z|+\rangle = |-\rangle$$

$$x|0\rangle = |1\rangle$$

$$z|-\rangle = |+\rangle$$

$$x|1\rangle = |0\rangle$$



$$A_s = x_1 x_2 x_3 x_4$$

$$A_s^2 = I$$

$$\prod_s A_s = I$$

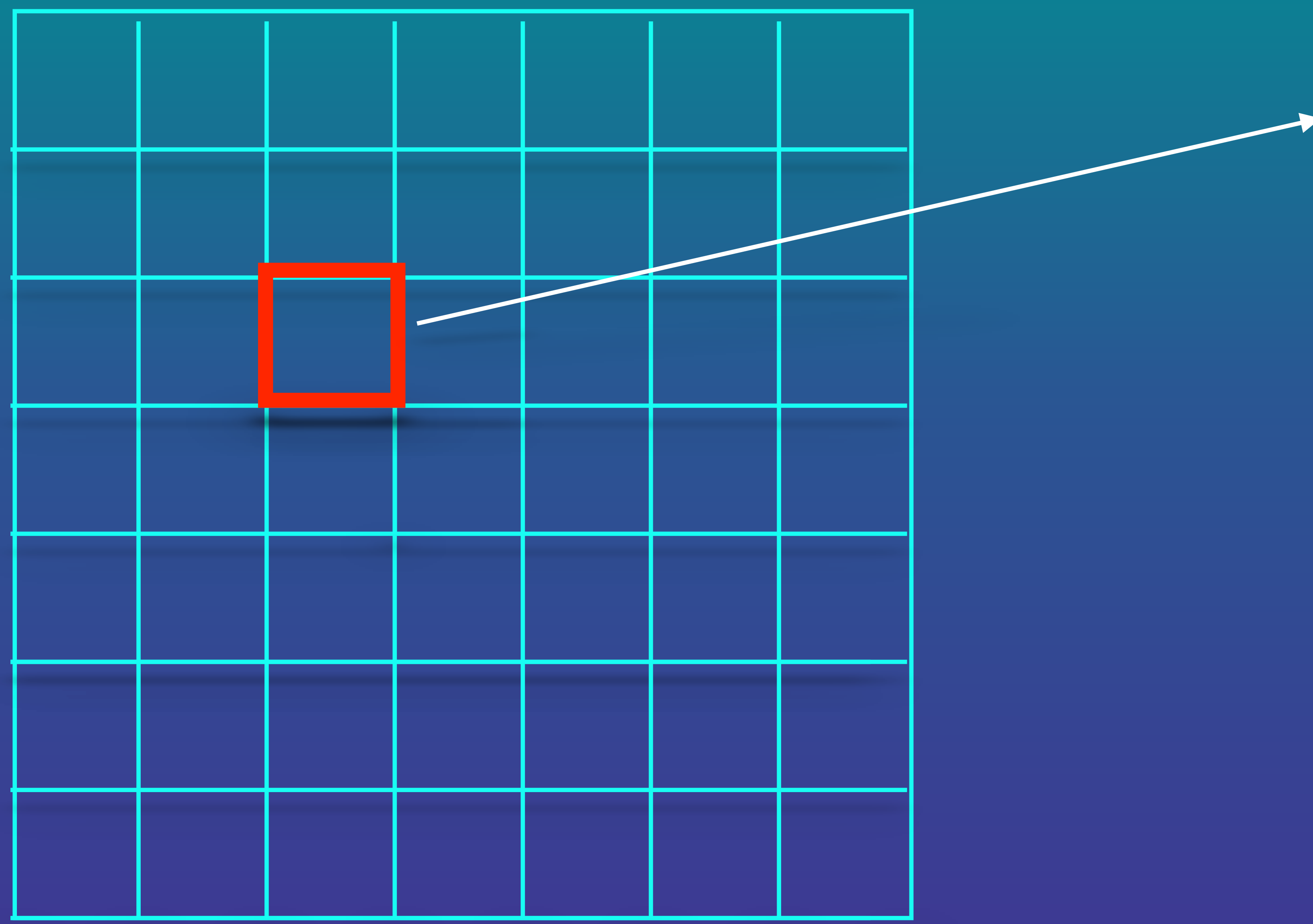
Number of vertices=N

Number of links=2N

Number of independent A's = N-1

$$Dimension\ \ of\ \ Hilbert\ Space = 2^{2N}$$

$$Degeneracy = \frac{2^{2N}}{2^{N-1}} = 2^{N+1}$$



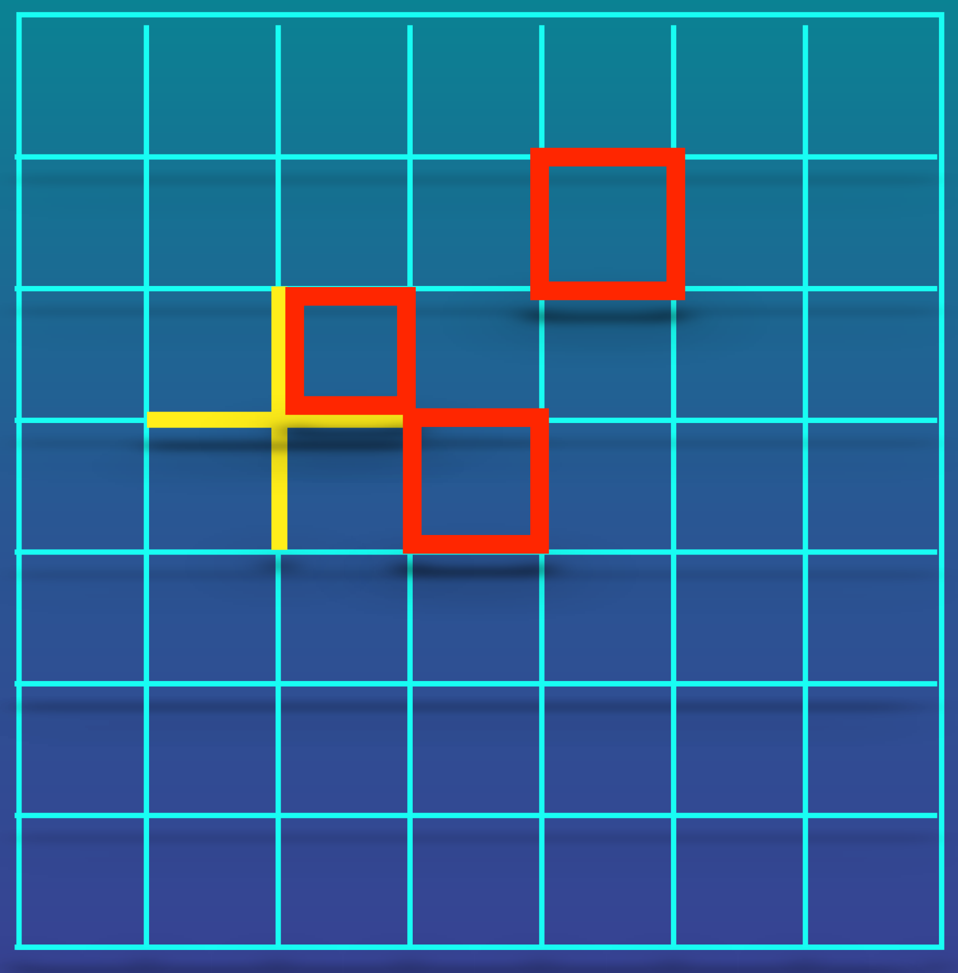
$$B_p = z_1 z_2 z_3 z_4$$

$$B_p^2 = I$$

$$\prod_p B_p = I$$

Number of faces = N

Number of Independent B’s = N-1



$$\left[A_s, B_p\right] = 0$$

$$H = -\sum_s A_s - \sum_p B_p$$

$$Degeneracy = \frac{2^{2N}}{2^{2N-2}} = 4$$

The ground state

$$H = -\sum_s A_s - \sum_p B_p$$

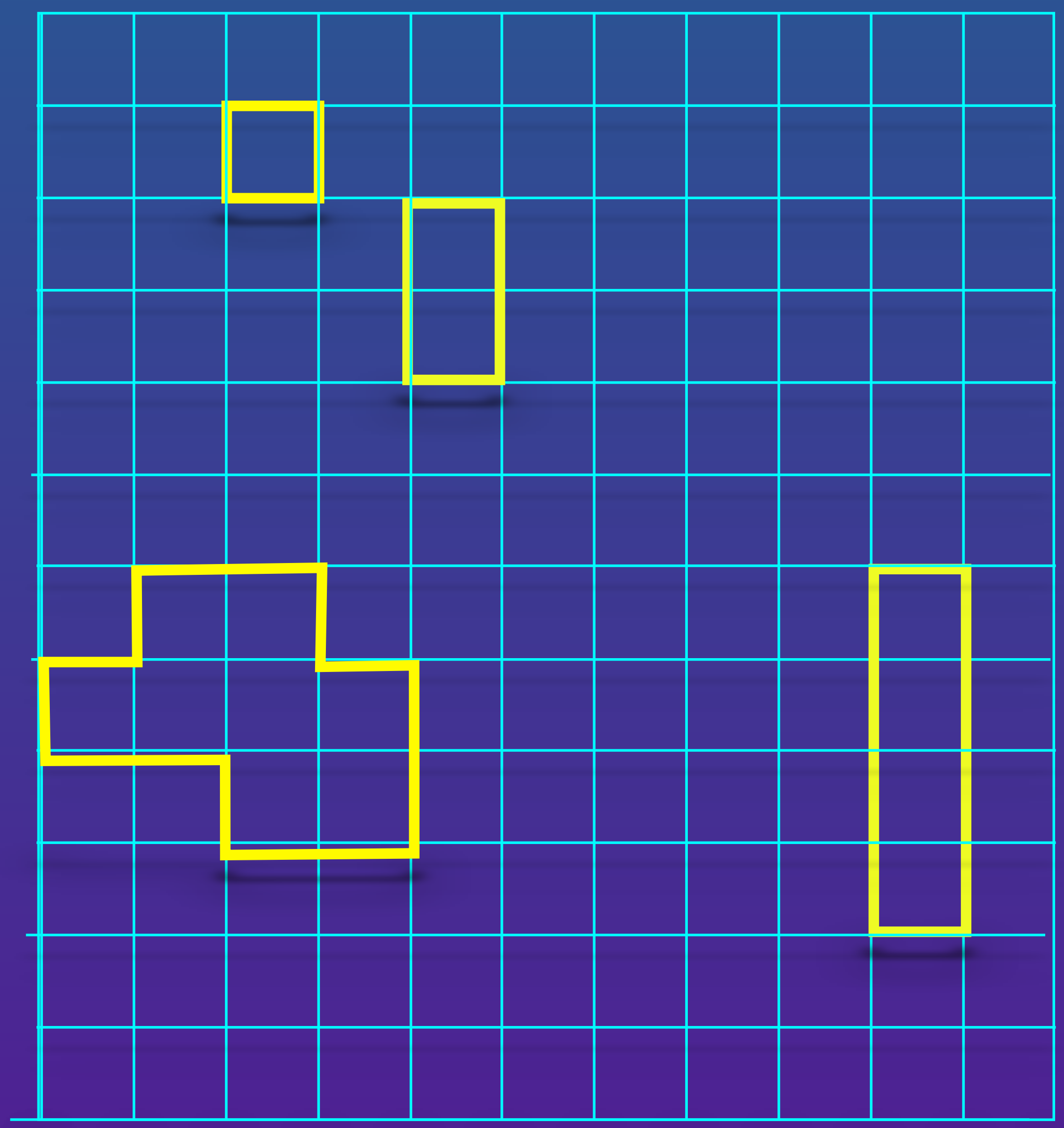
$$A_s|\phi\rangle = |\phi\rangle \qquad B_p|\phi\rangle = |\phi\rangle$$

How the ground state looks like?

$$|\Omega\rangle = |+\rangle^{\otimes N}$$

$$|\varphi_0\rangle = \prod_p (1 + B_p)|\Omega\rangle$$

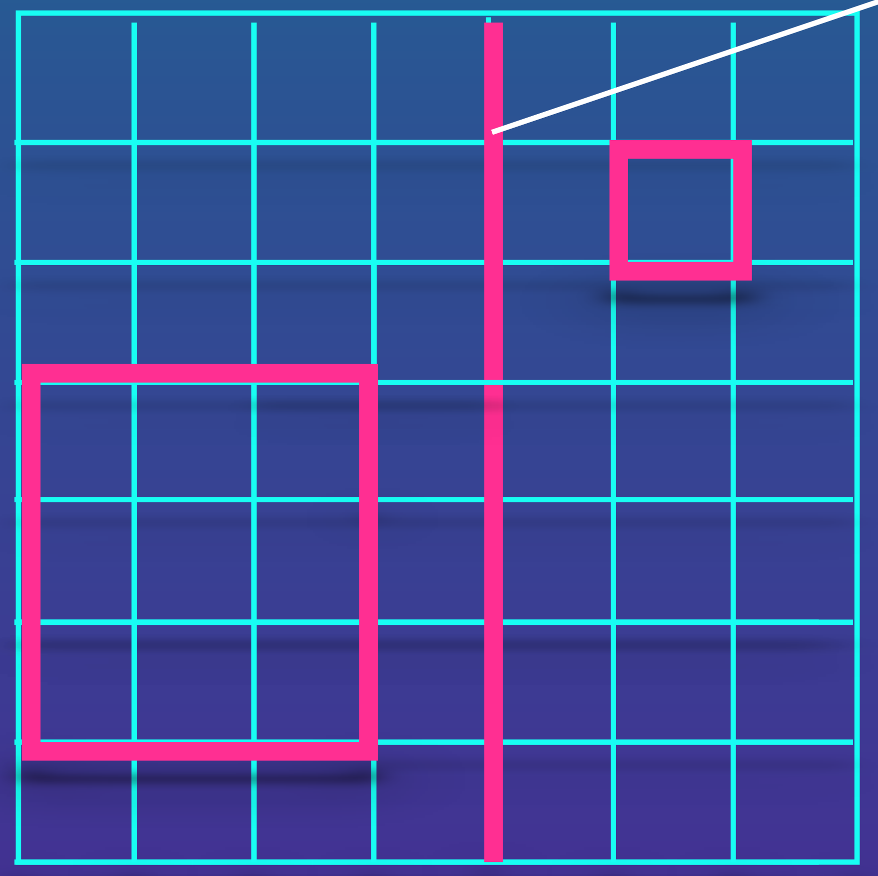
$$A_s|\Omega\rangle = |\Omega\rangle$$



$$z|+\rangle = |-\rangle$$

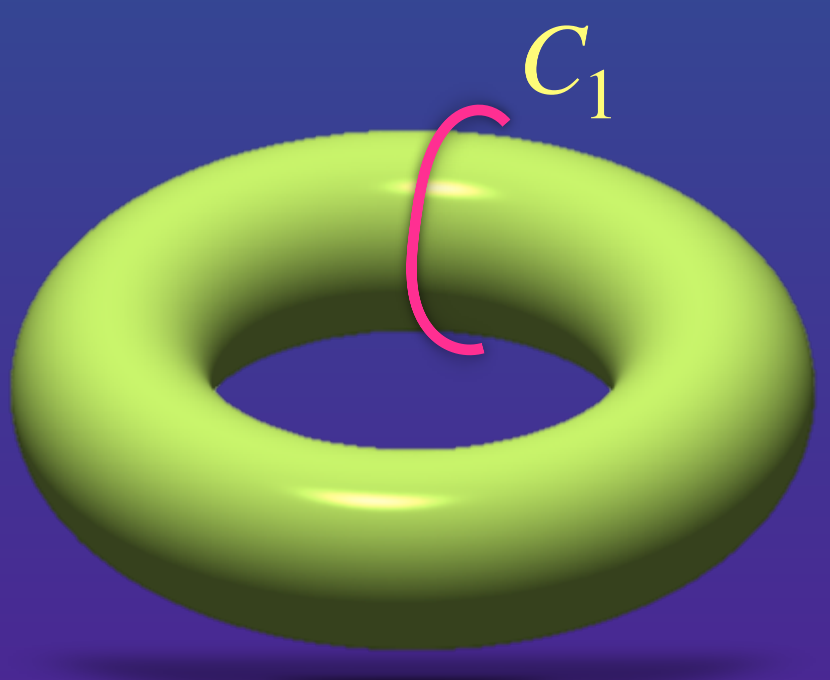
$$B_p|\Omega\rangle \neq |\Omega\rangle$$

String operators which create degenerate states

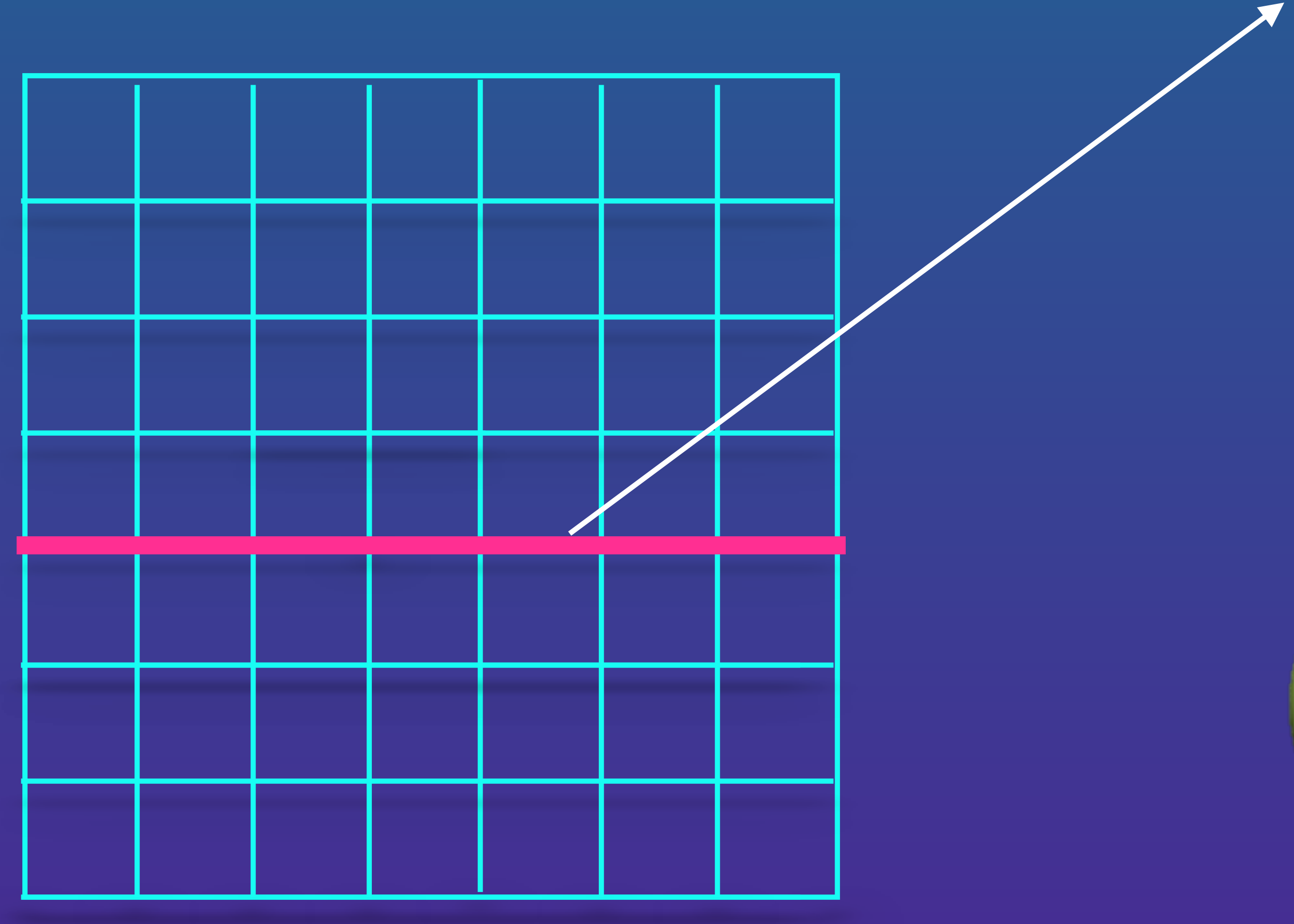


$$X_1 = \prod_{i \in C_1} z_i$$

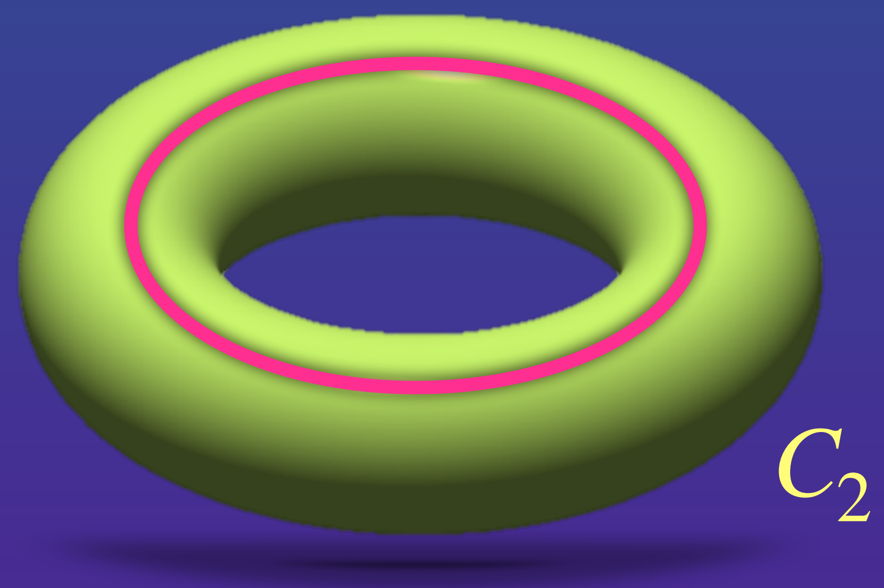
$$[X_1, H] = 0$$



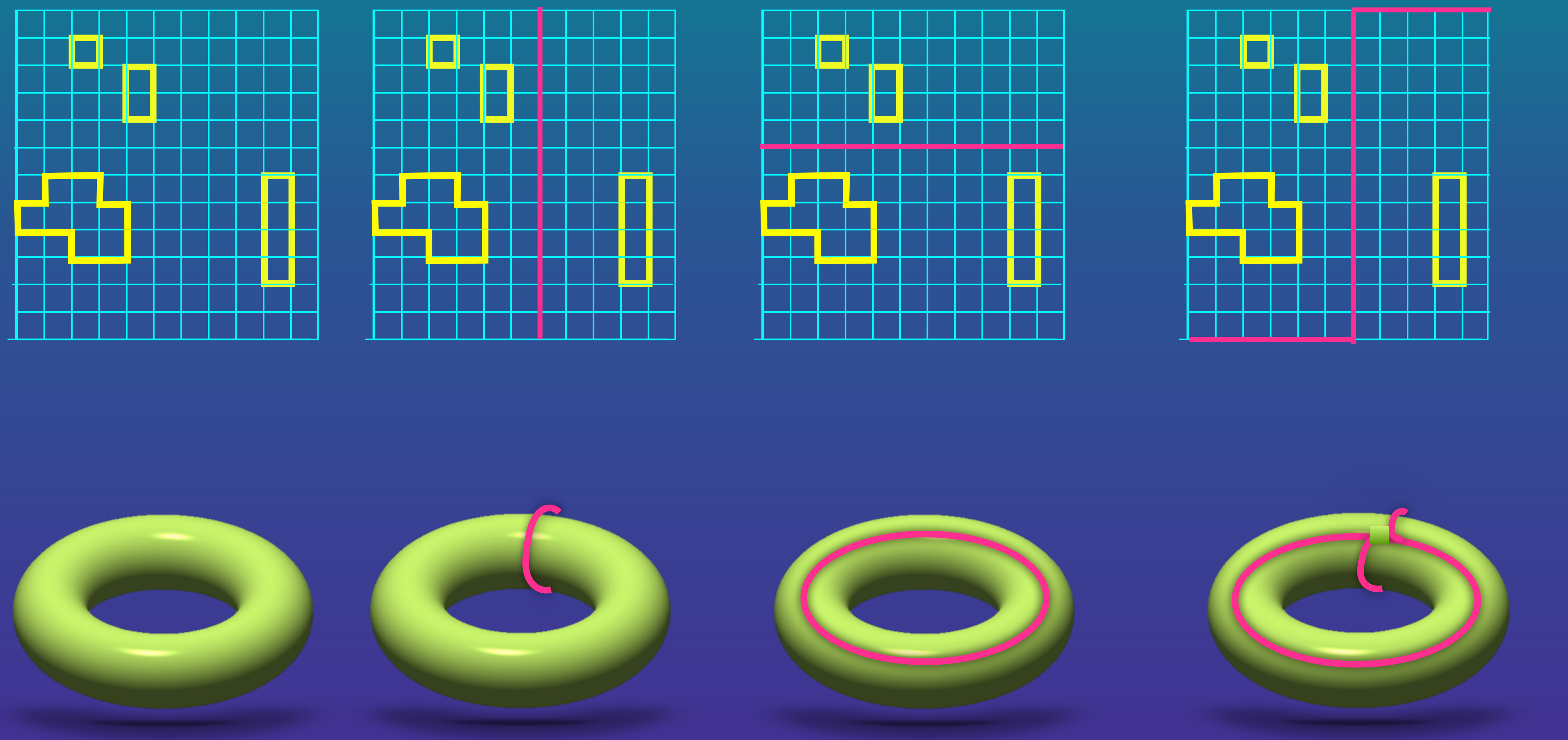
$$X_2 = \prod_{i \in C_2} z_i$$



$$[X_2, H] = 0$$

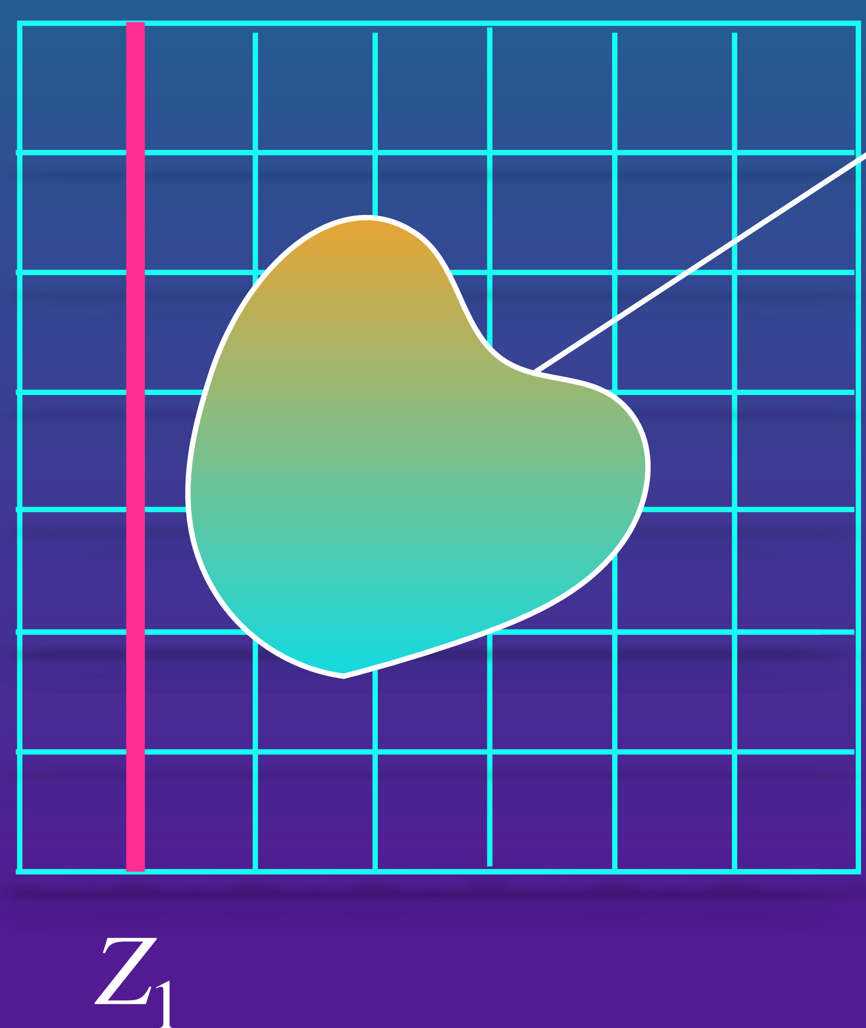


Four ground states



$|\phi_{00}\rangle$ $\quad$ $|\phi_{10}\rangle = X_1|\phi_{00}\rangle$ $\quad$ $|\phi_{01}\rangle = X_2|\phi_{00}\rangle$ $\quad$ $|\phi_{11}\rangle = X_1X_2|\phi_{00}\rangle$

Local operators cannot distinguish these four ground states.



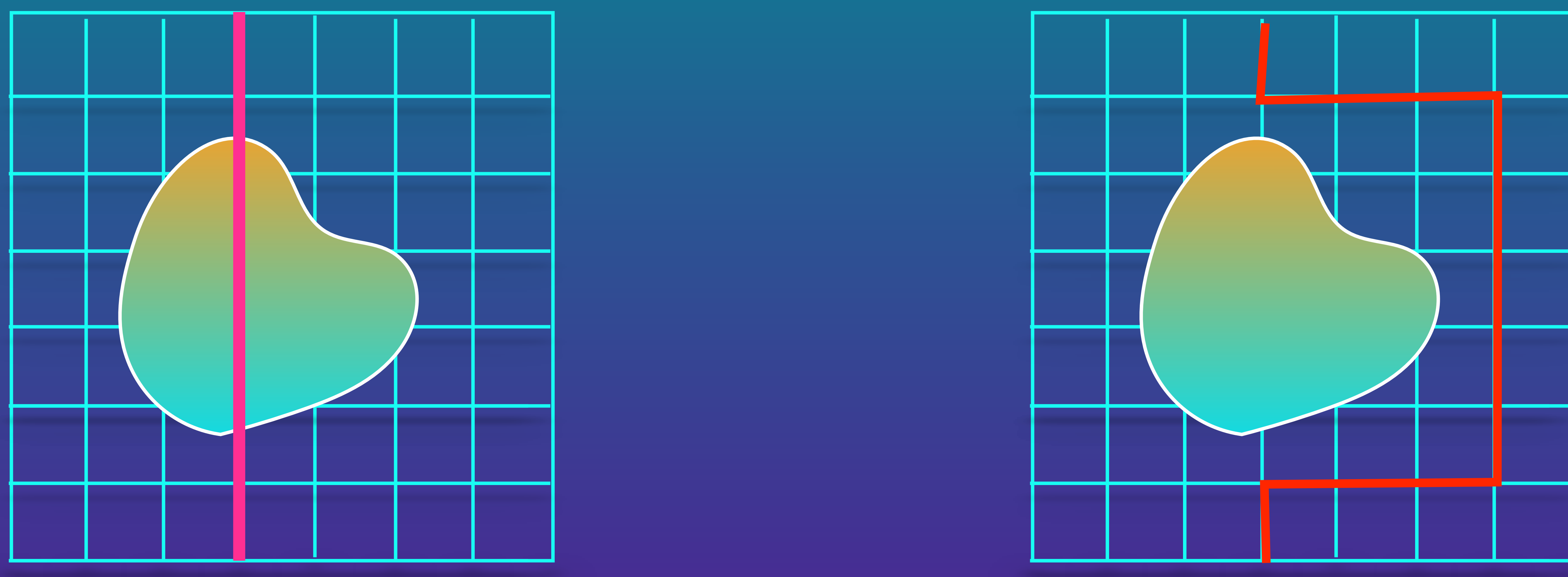
The support of an arbitrary local operator

O = the local operator

It is obvious that $X_1 O = O X_1$

$$\langle \phi_{10} | O | \phi_{10} \rangle = \langle \phi_{00} | X_1 O X_1 | \phi_{00} \rangle = \langle \phi_{00} | O | \phi_{00} \rangle$$

What happens in this case?

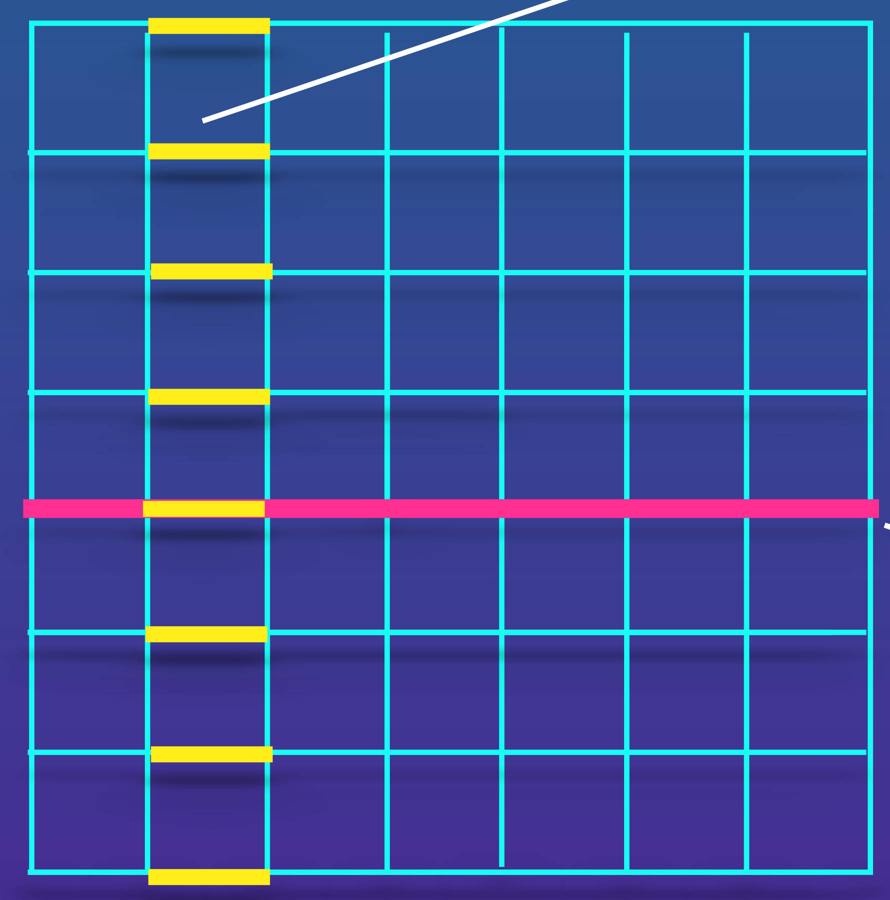


X_1 X_1

Sine the operator is local, we can deform the line:

Again: $X_1 O = O X_1$

String operators which distinguish the states!

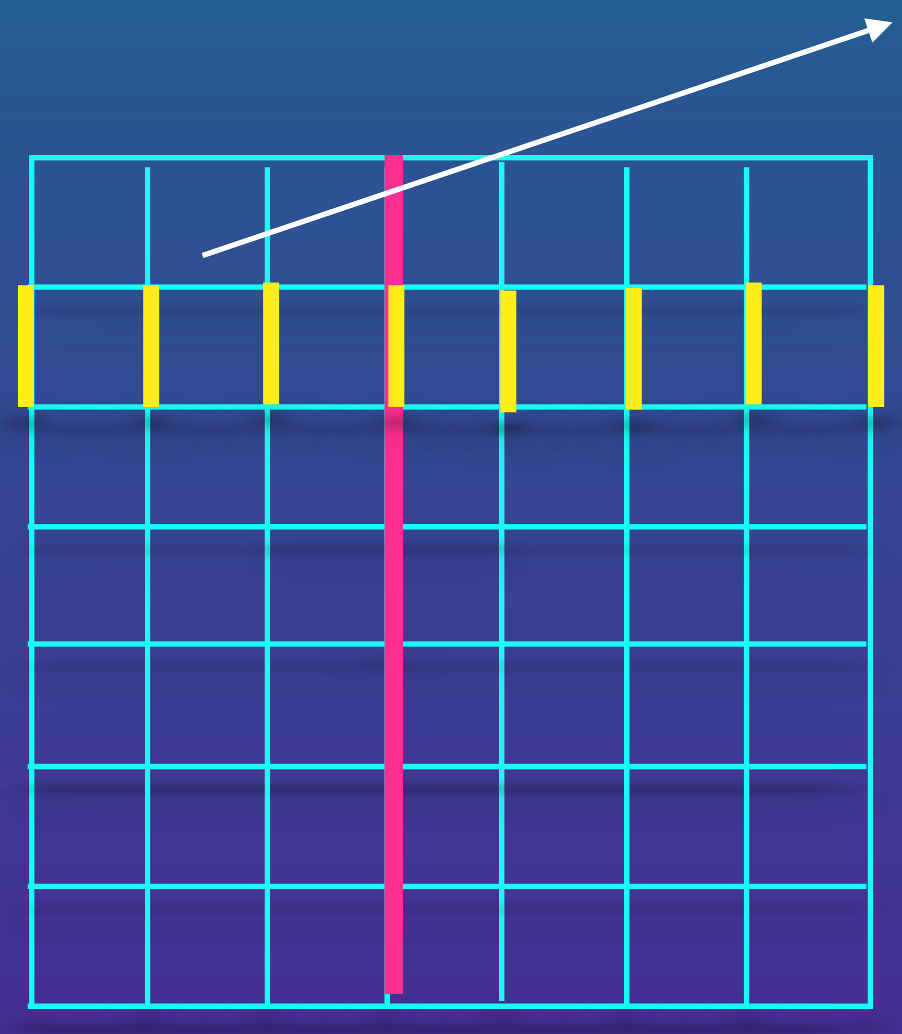


$$Z_2 = \prod_{i \in C_1} x_i$$

$$[Z_2, H] = 0$$

$$X_2 = \prod_{i \in C_2} z_i$$

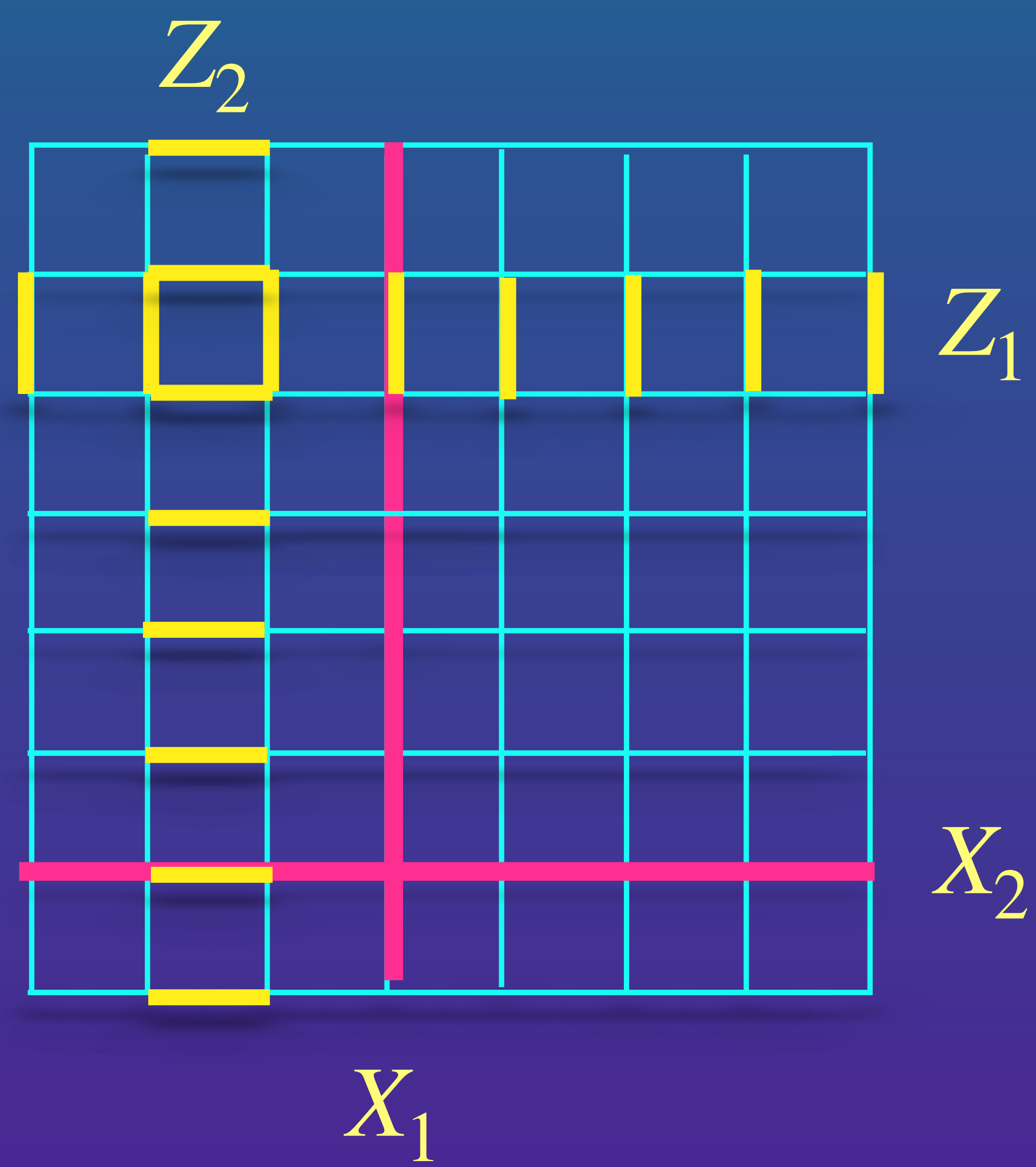
String operators which distinguish the states!



$$Z_1 = \prod_{i \in C_2} x_i$$

$$[Z_1, H] = 0$$

Summary

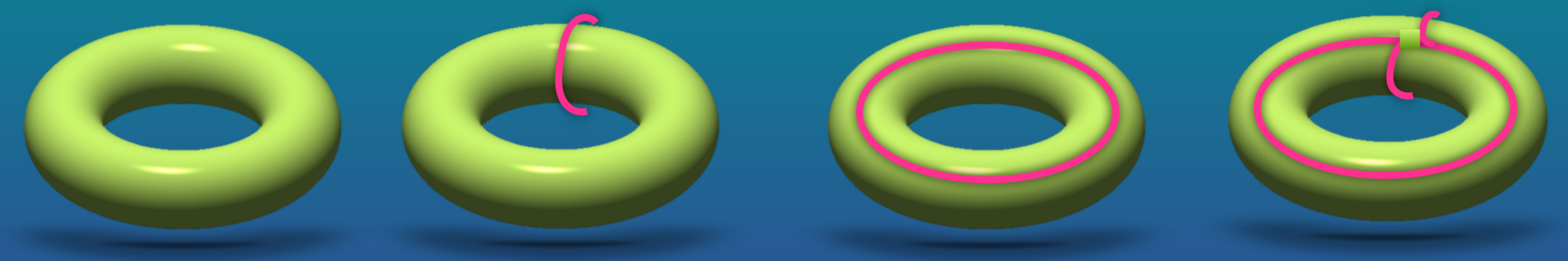


$$Z_1X_1 = -X_1Z_1$$

$$Z_2X_2 = -X_2Z_2$$

$$Z_1X_2 = X_2Z_1$$

$$Z_2X_1 = X_1Z_2$$



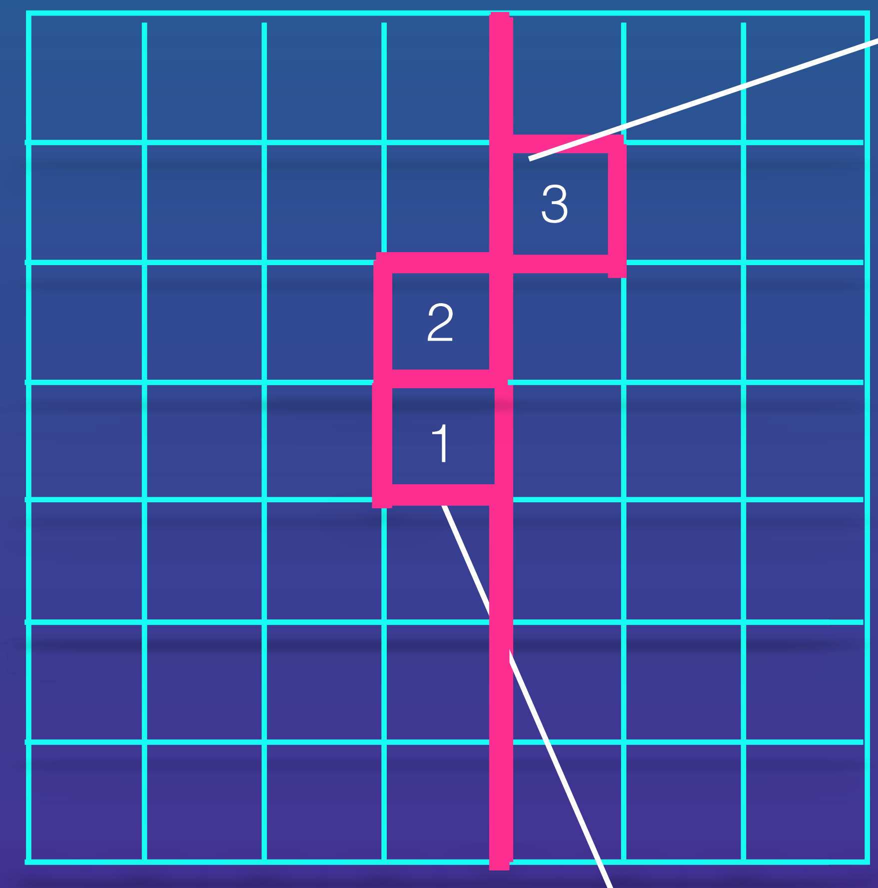
	$\lvert\phi_{00}\rangle$	$\lvert\phi_{10}\rangle = X_1\lvert\phi_{00}\rangle$	$\lvert\phi_{01}\rangle = X_2\lvert\phi_{00}\rangle$	$\lvert\phi_{11}\rangle = X_1X_2\lvert\phi_{00}\rangle$
Z_1	1	–1	1	–1
Z_2	1	1	–1	–1

Why degeneracy is not removed by local perturbations?

$$\Delta E_\alpha = \left\langle \psi_\alpha \middle| \sum_i O_i \middle| \psi_\alpha \right\rangle$$

$$\Delta E_\alpha = \Delta E_\beta$$

Where is Topology?

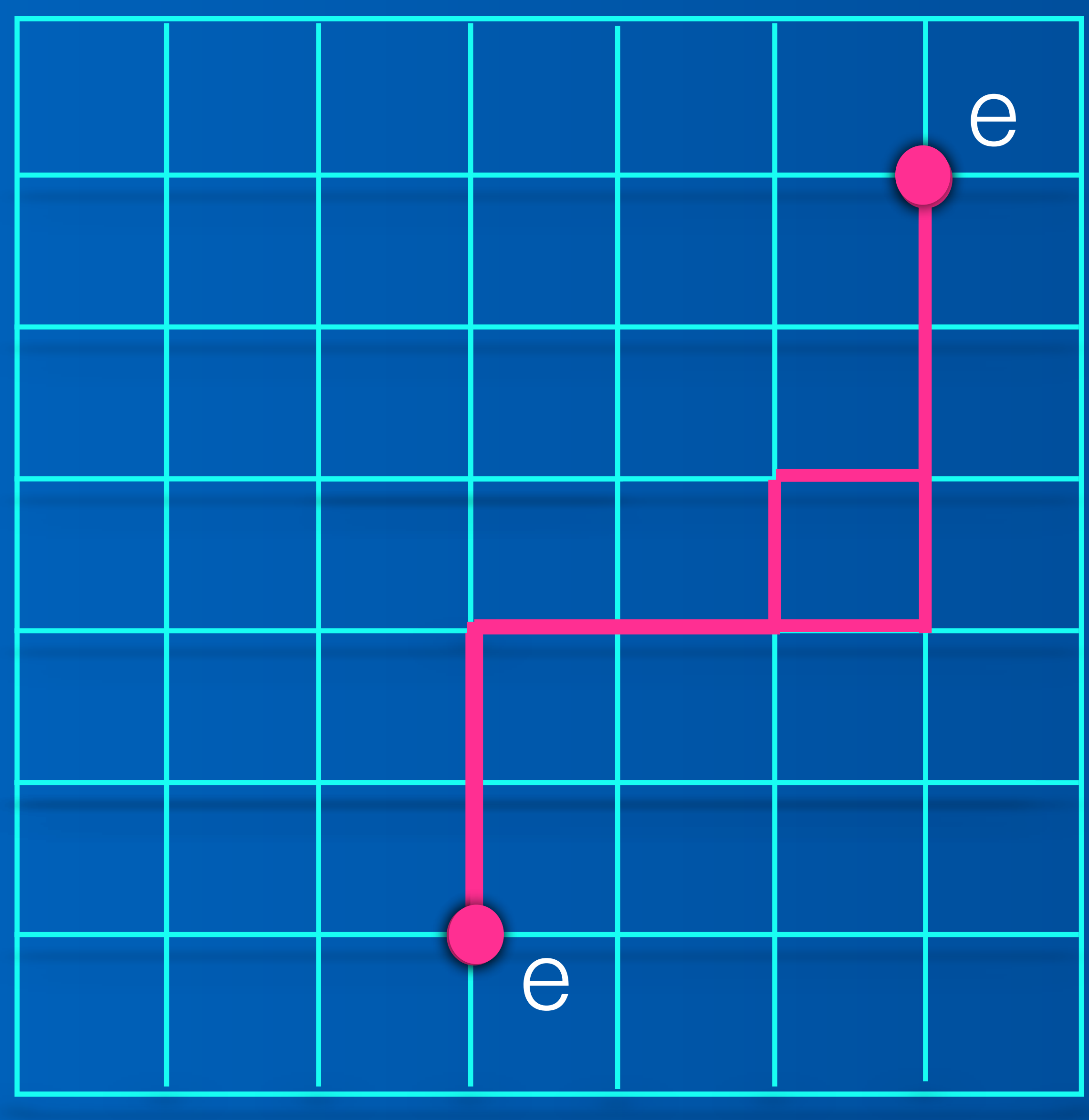


$$X_1 = \prod_{i \in C_1} z_i$$

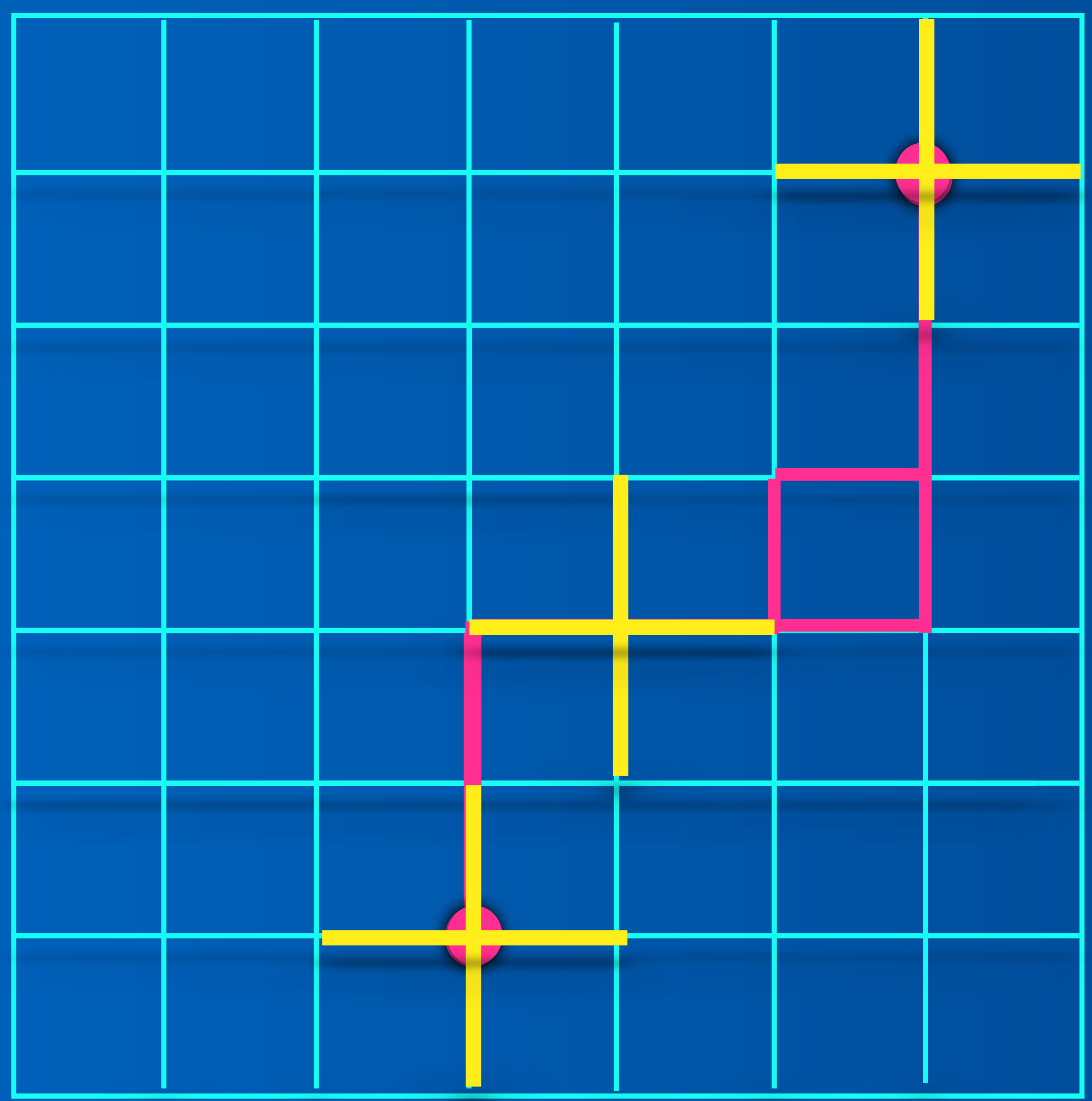
$$X_1' = X_1 B_1 B_2 B_3$$

Degeneracy depends on topology

Excited States: 1- Electric Anyons.



Excited States: 1- Electric Anyons.



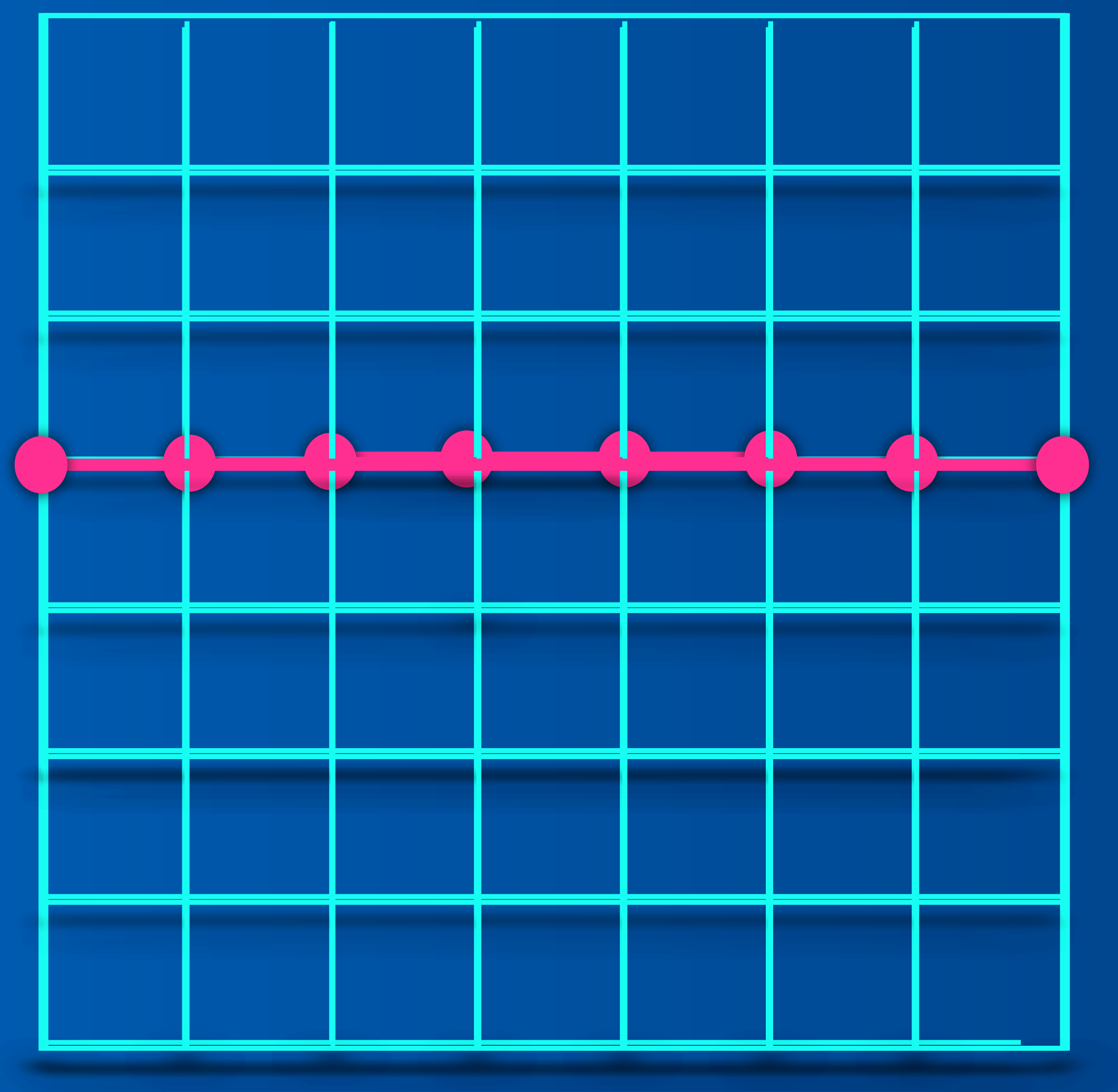
Each Anyon has an energy of

2 units.

Anyons are created in pairs.

The energy of the pair doesn't depend on the path connecting them.

Another interpretation of String Operators



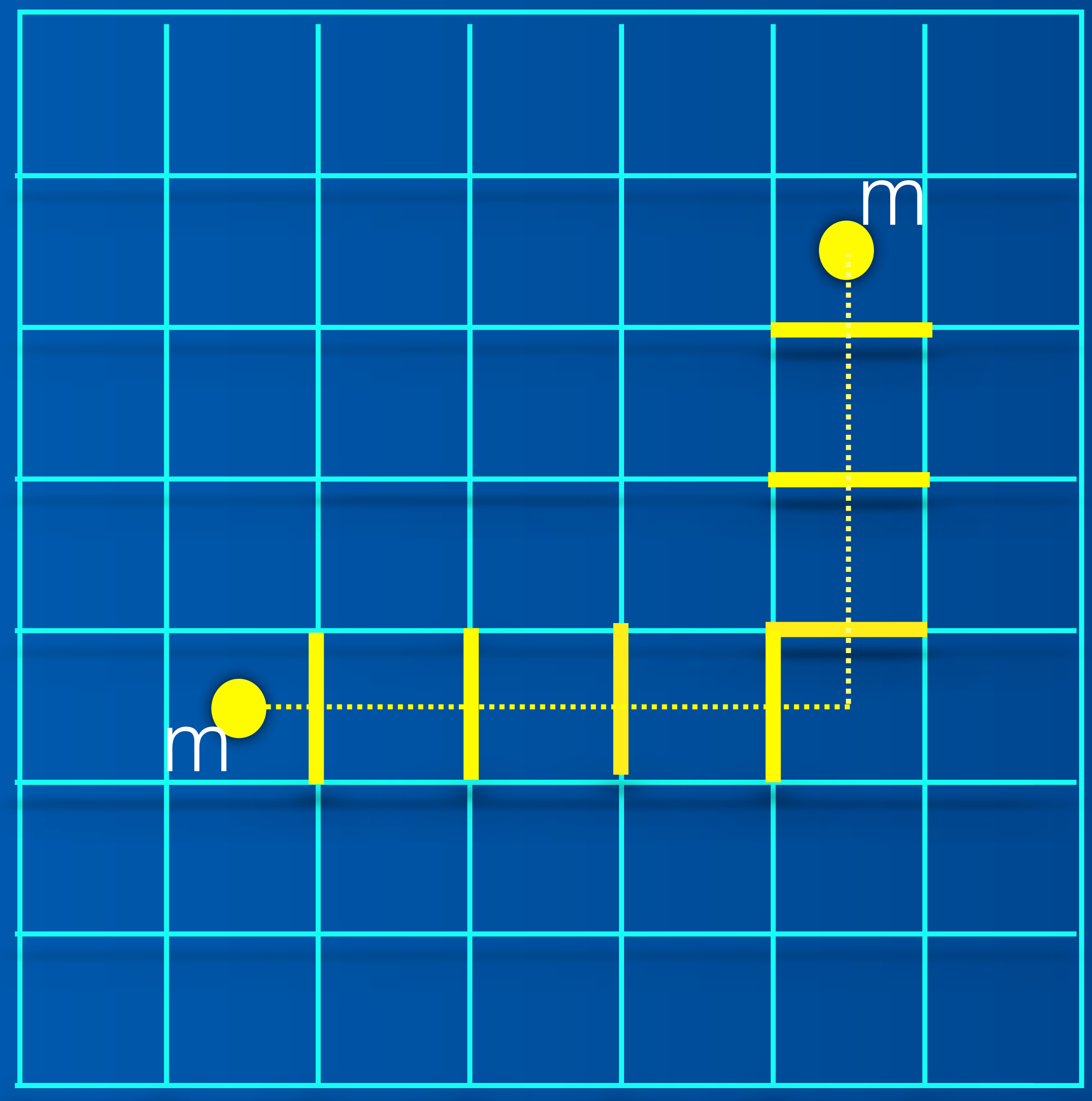
So by creating two electric Anyons,

Moving them across the Torus,

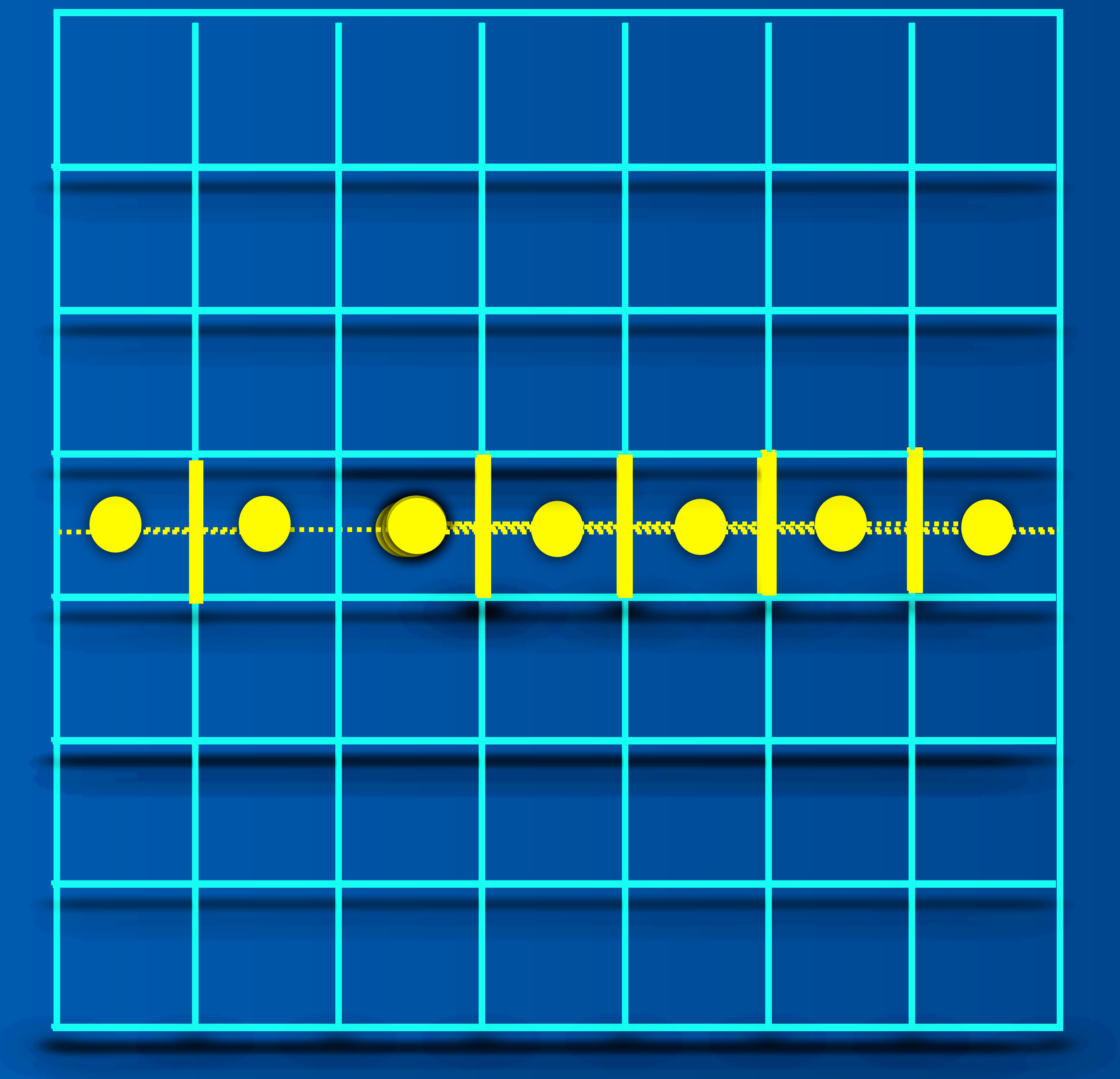
And annihilating them in the end,

We can implement a X gate on either of the qubits.

Excited States: Magnetic Anyons



Another interoperation of string operators



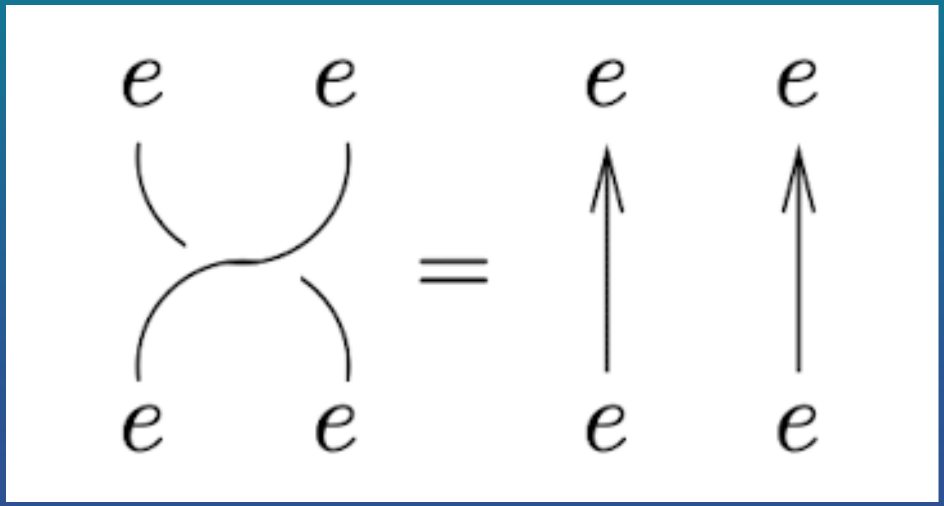
So by creating two Magnetic Anyons,

Moving them across the Torus,

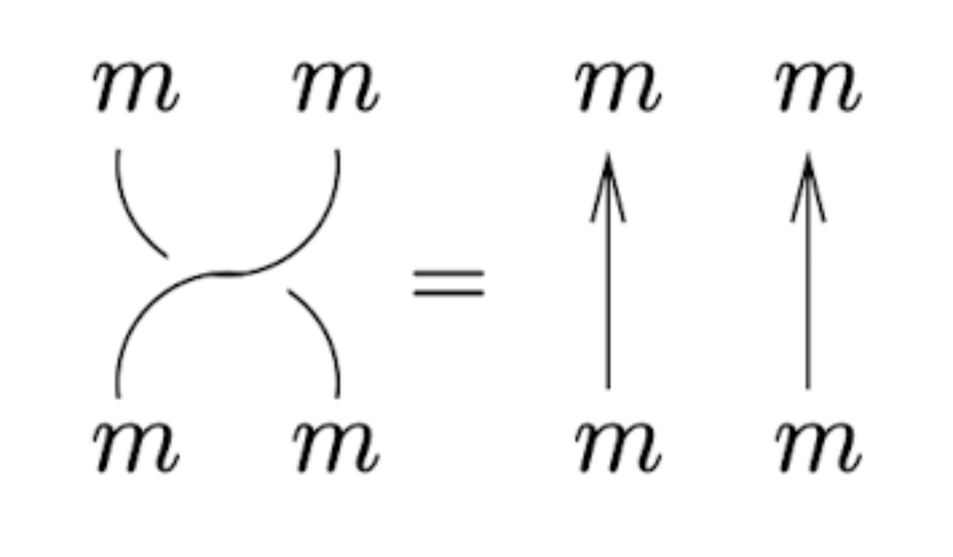
And annihilating them in the end,

We can implement a Z gate on either of the qubits.

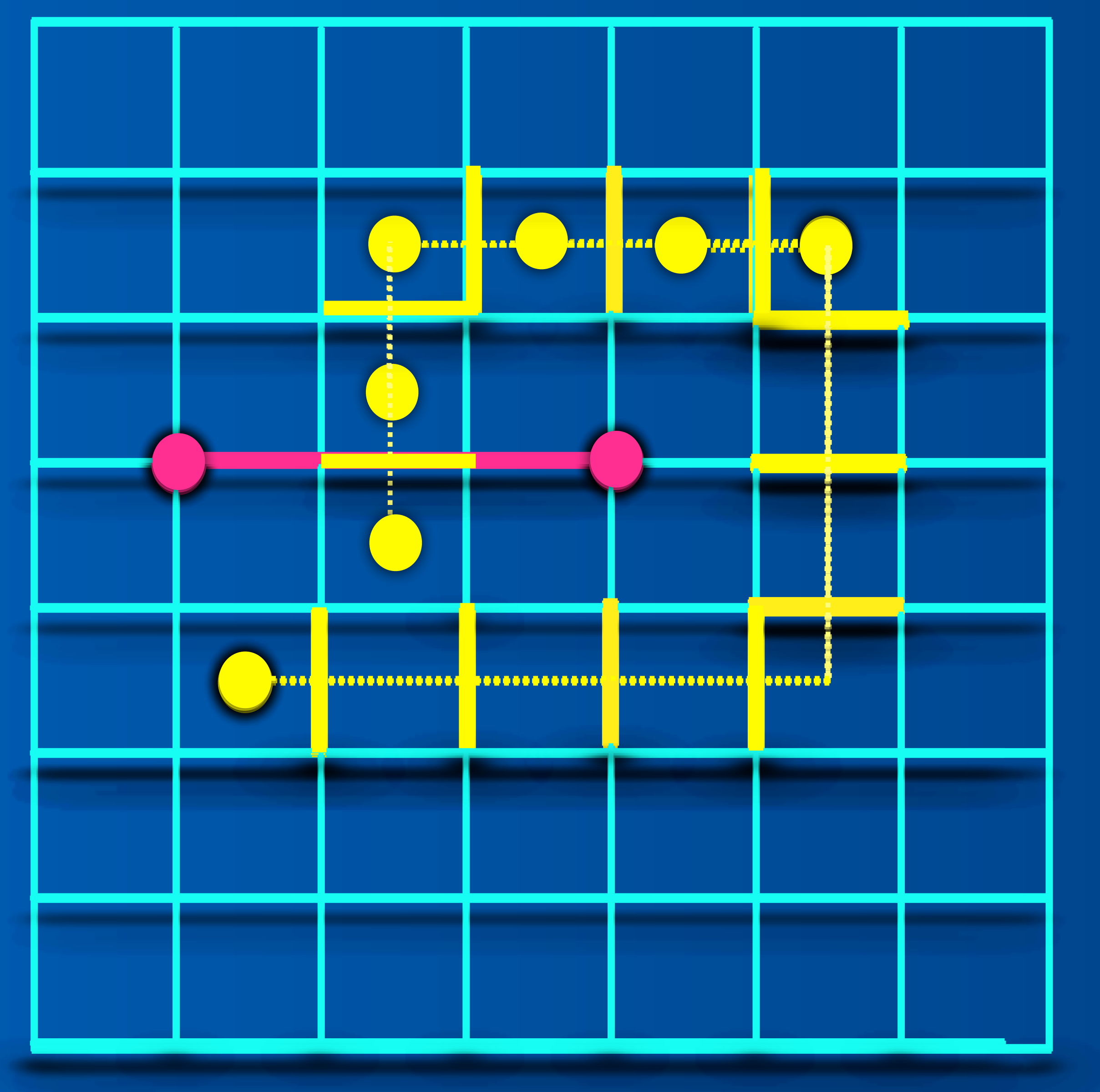
Electric excitations behave as Bosons with respect to each other.



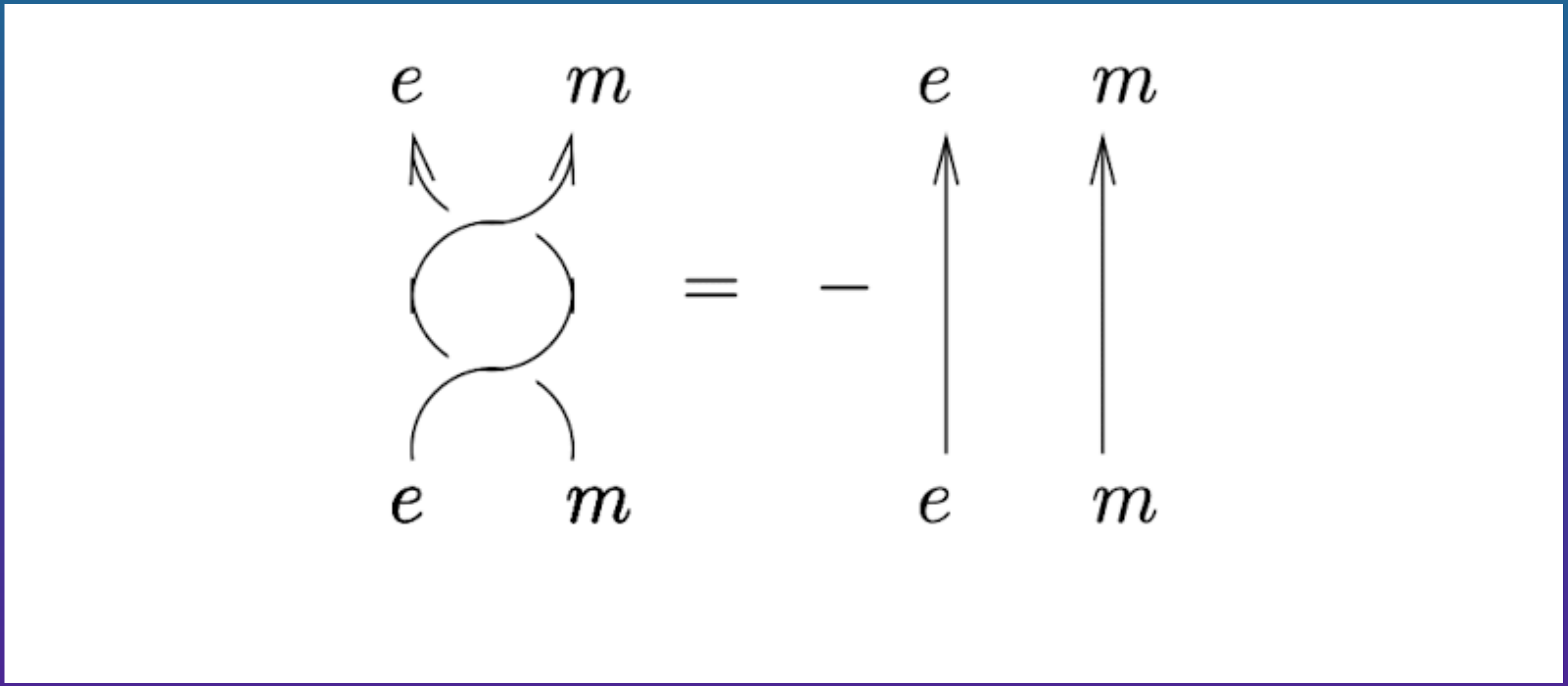
Magnetic excitations behave as Bosons with respect to each other.



Why These are Anyons?

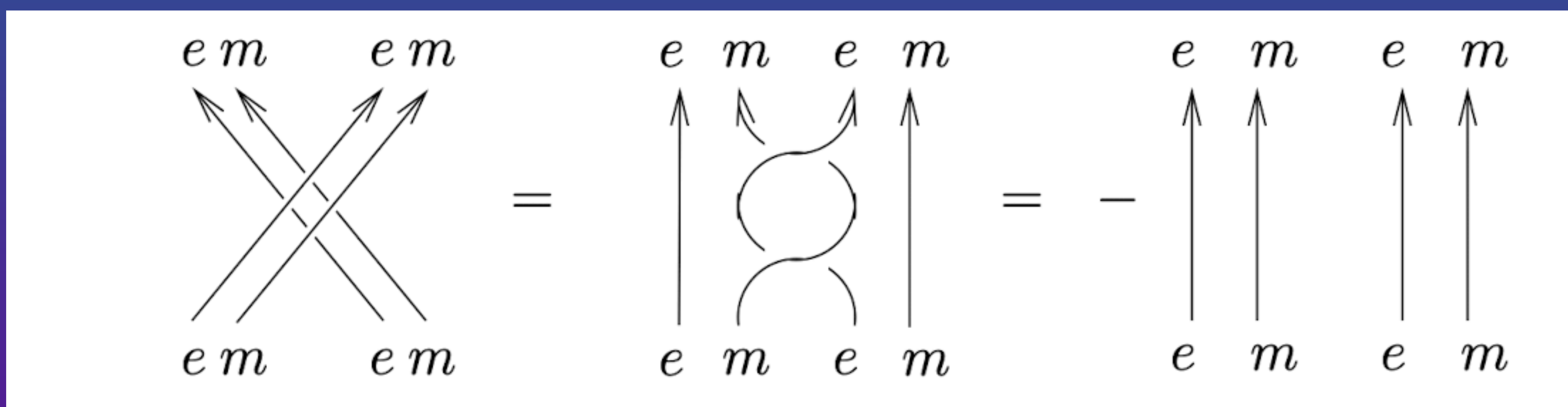


But Electric and Magnetic excitations behave as Fermions with respect to each other.



But e and m are not identical particles.

The pair (em) behaves as a fermion.



Fusion Rules of Toric Code Anyons

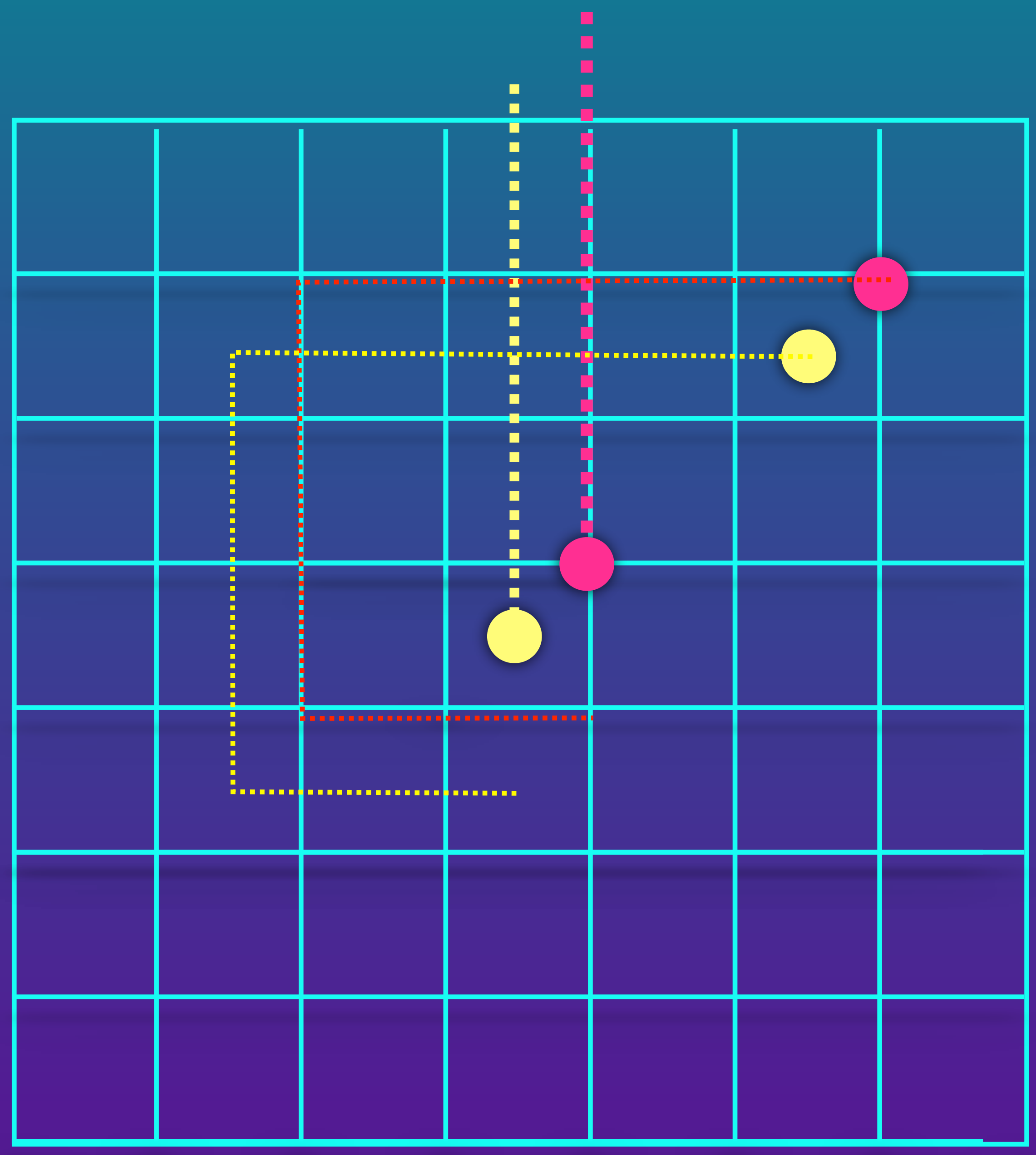
$$\{1,\ e,\ m,\ \epsilon\}$$

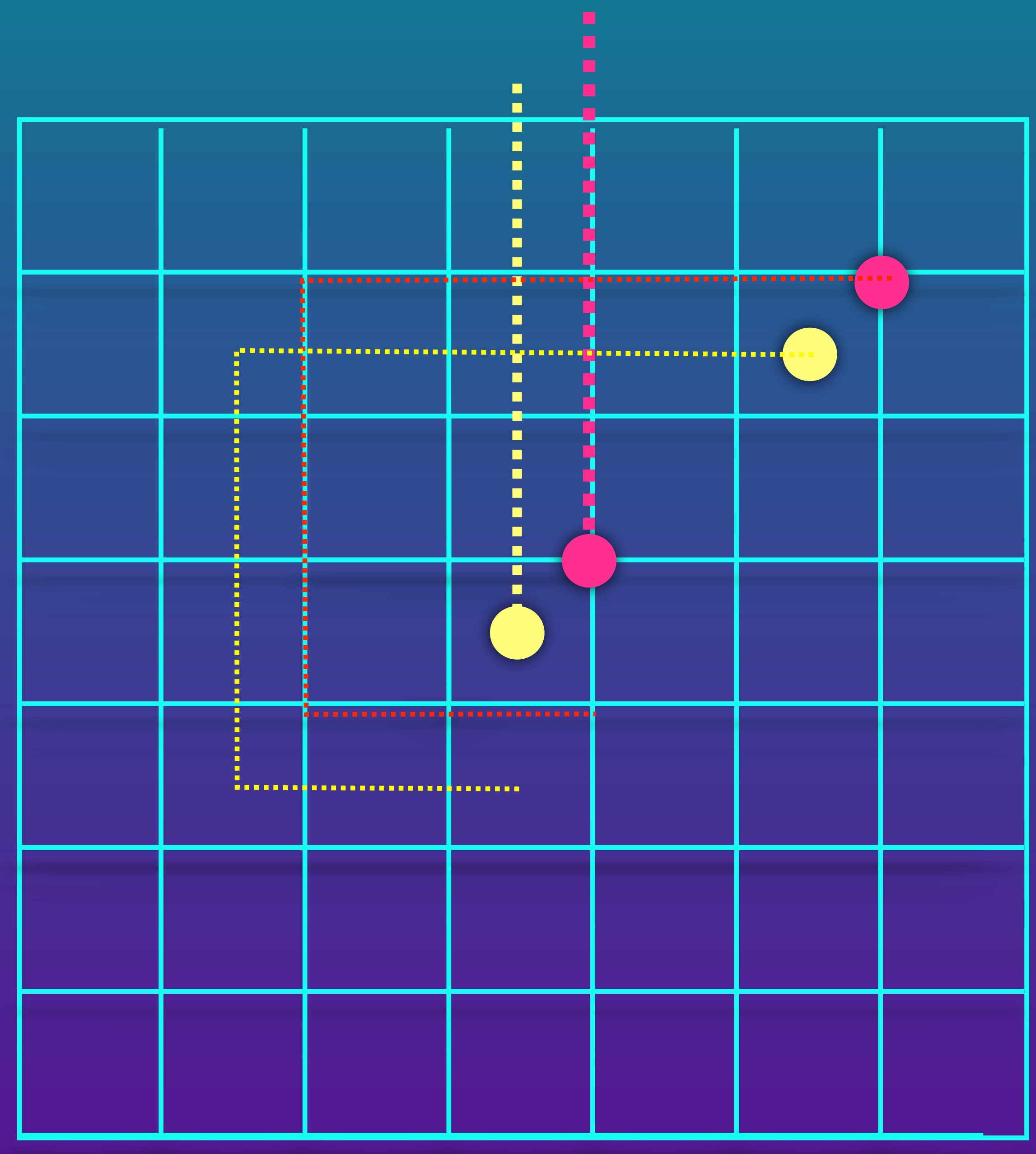
$$1 \times a = a$$

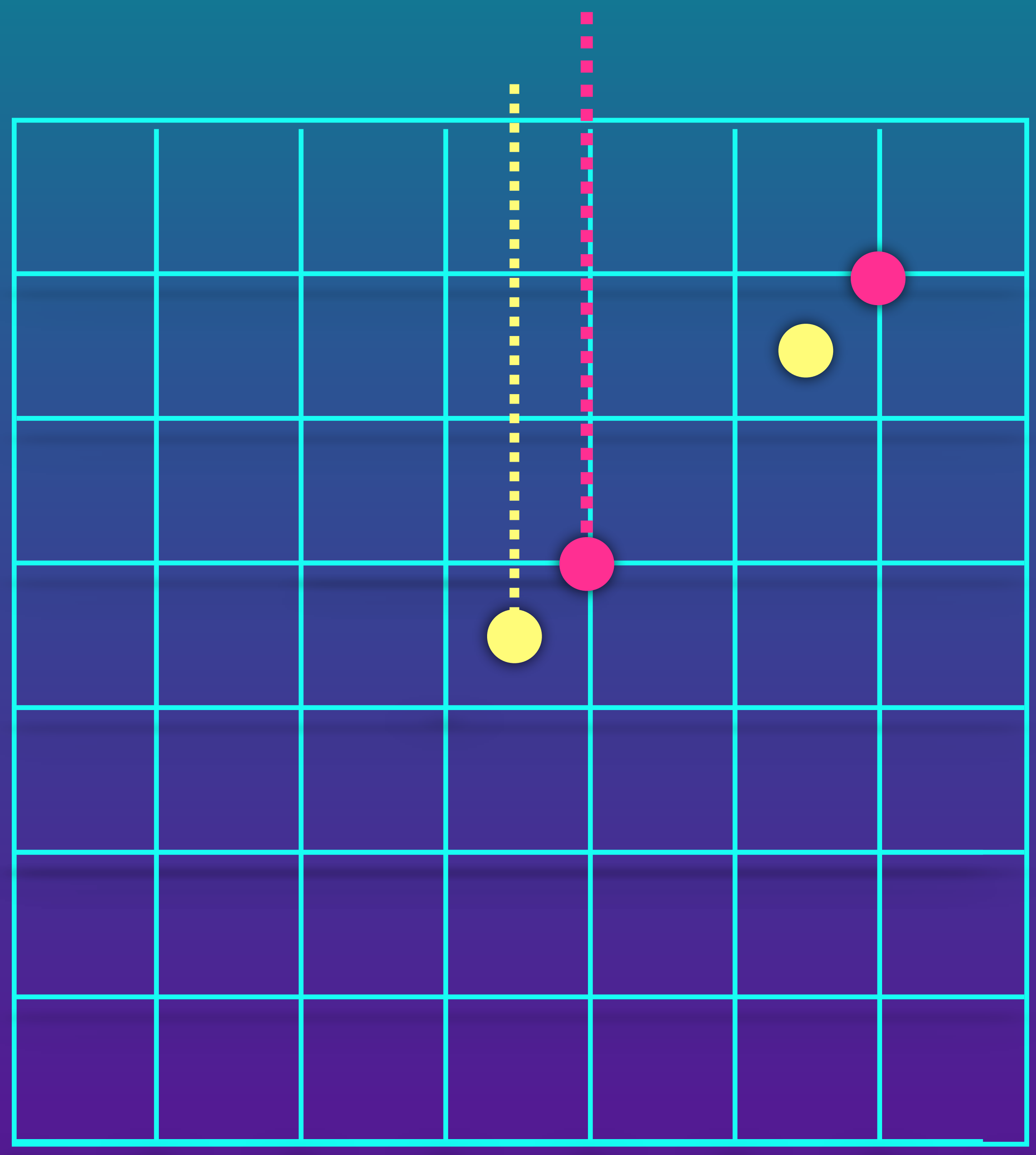
$$e \times e = 1 \qquad m \times m = 1$$

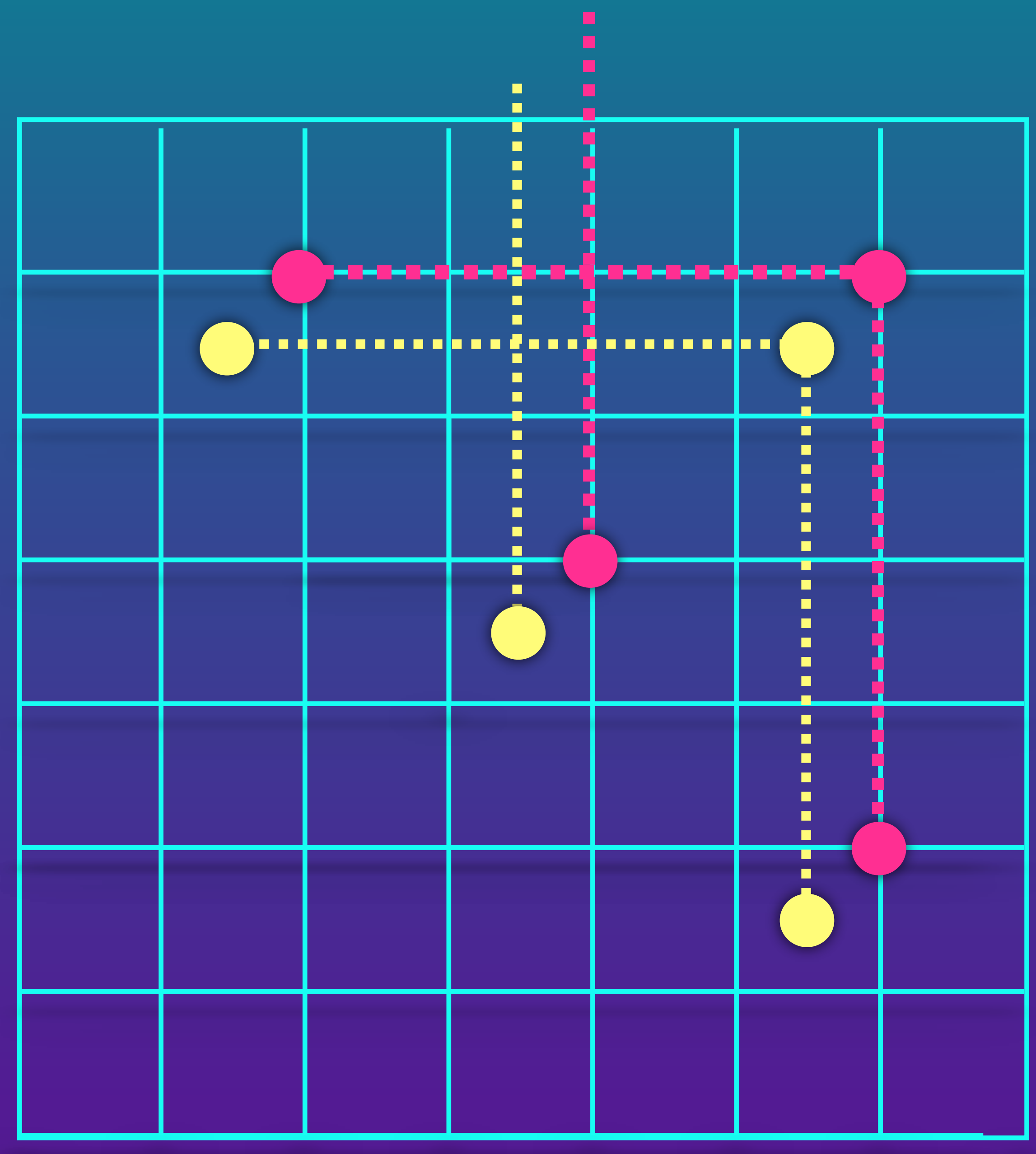
$$e \times m = \epsilon$$

$$\epsilon \times e = m \qquad \epsilon \times m = e$$









So we can do simple,

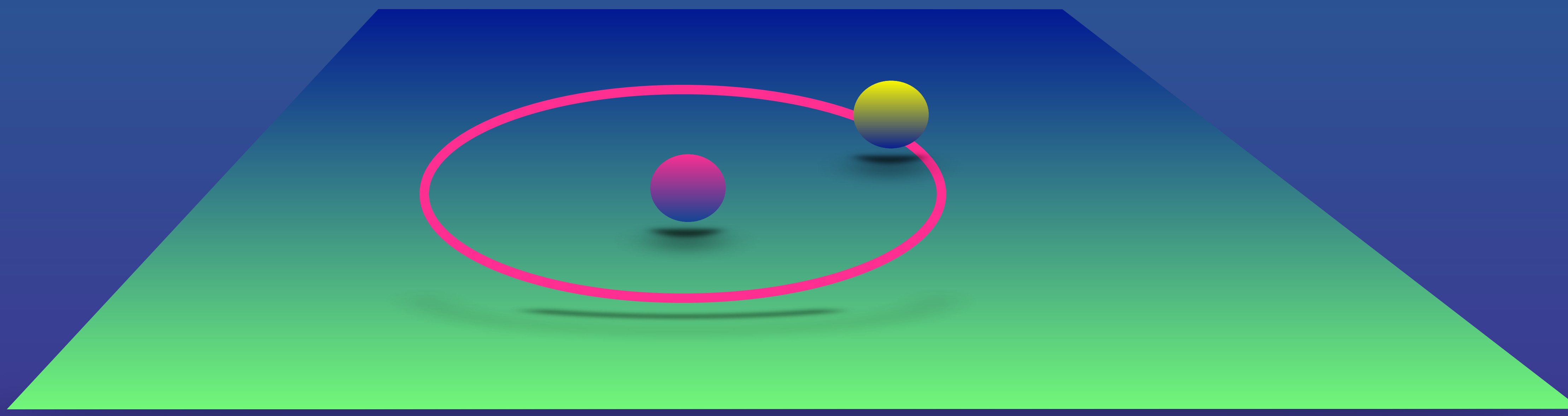
X, Z and Y gates

in a fault-tolerant way.

Unfortunately
the Abelian Models
are not Universal.

We have to consider
Non-Abelian Models.

Non-Abelian Anyons



$$|\psi_i\rangle \rightarrow U_{ij}|\psi_j\rangle$$

End of part I